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# Analysis of Isotropic Beams Using Method of Initial Functions (MIF)

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ABSTRACT: In this paper, method of initial functions (MIF) has been used to analyse the isotropic beams. The method of initial function is an analytical method of elasticity theory. No assumptions regarding the distribution of stress or displacements are needed in this method. This method has applications in various fields of structural engineering such as plates, shells and beams. The equations of two dimensional elasticity have been used for deriving the governing equations. Numerical solutions of the governing equations have been presented for simply supported beam loaded with uniformly distributed load. Two cases with varying depth to span ratio have been considered for analysis and compared the results with the bending theory.

KEYWORDS: Beams, method of initial functions, stress and displacement.

## 1 INTRODUCTION

The theories which are based on assumptions regarding the distribution of stresses and displacements are of a practical utility in the case of those problems, where the beam thickness is small. The results obtained by these theories are away from actual physical behavior of flexural members. The two major theories generally used for the beam analysis, The Bernoulli-Euler theory and Timoshenko beam theory, are based on assumptions. One common assumption is that transverse sections which are plane before bending remain plane after bending. However, it has been observed that beam sections especially in the case of deep beams, warp under loaded conditions. So in the problems involving thick beams and layered beams, it becomes difficult to obtain useful results using these theories.

An alternative approach, used in this paper, for the analysis of isotropic beam is method of initial functions. Method of Initial Functions in short called MIF. It is an analytical method of elasticity theory. The method makes it possible to obtain exact solutions of different types of problems, i.e., solutions without the use of hypotheses about the character of stress and strain. According to this method, the basic desired functions are the displacements and stresses.The system of differential equations are obtained from equations of Hook's law and equilibrium equations by replacing stresses by the displacements according to elasticity relations. The order of the derived equations depends on the stage at which the series representing the stresses and displacements are truncated.

From the literature it has been observedthat this method has various applications in structural engineering butvery few researchers have used MIF for analysis of beams. Vlasov(1957) suggested a method for solving problems of theory of elasticity for the analysis thick plates as well as shells which is known as the method of initial functions.The method of initial function has been applied (Iyengar and Pandya 1986) for deriving higher order theories for laminated composite thick rectangular plates. Iyengar(1974, 1979) used method of Initial functions for the analysis of rectangular and long beams. (Dubey 2005) applied method of initial functions for the analysis of orthotropic deep beams. Patel (2012) with the help of MIF developed governing equation for the study of composite beams having two layers of orthotropic material.

There so many other theories which are used for the analysis of beams. (Ghugal and Sharma, 2011) Developed Hyperbolic Shear Deformation Theory for transverse shear deformation effects. It is used for the static flexure analysis of thick isotropic beams and the results of the present theory are compared with those of other refined shear deformation theories of beam. A layer wise trigonometric shear deformation theory is used (Ghugal and Shinde 2013) for the analysis of two layered cross ply laminated simply supported and fixed beams subjected to sinusoidal load. Virtual work principle is employed to obtain governing equations and boundary conditions.

The Ant Colony Optimization was applied to the optimization of laminated composite plates, (Luersen et al. 2013). The formal representation that the artificial ants use in a problem formulation for ACO was applied to the analysis of laminated composite plates. (Gao and Wang 2006) shown that the displacements and stresses of the beam can be represented by the angle of rotation and the deflection of the neutral surface. Based on the refined beam theory, the exact equations for the beam without transverse surface loadings are derived.

## 2 FORMULATION OF MIF

According to this method, the basic desired functions are the displacements and stresses, the system of differential equations which are obtained from equations of Hook's law and equilibrium equations by replacing stresses by the displacements according to elasticity relations. The order of the derived equations depends on the stage at which the series representing the stresses and displacements are truncated.

The equations of equilibrium for solids ignoring the body forces for two-dimensional case are:

$$\frac{\partial \sigma_x}{\partial x}+\frac{\partial \tau_{xy}}{\partial y}=0 \quad (1)$$

$$\frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \sigma_y}{\partial y}=0 \quad (2)$$

The stress-strain relations for isotropic material are:

$$\sigma_x = C_{11}\varepsilon_x + C_{12}\varepsilon_y \quad (3)$$

$$\sigma_y = C_{12}\varepsilon_x + C_{22}\varepsilon_y \quad (4)$$

$$\tau_{xy} = C_{33}\gamma_{xy} \quad (5)$$

The constants $C_{11}$ to $C_{33}$ expressed in terms of the elastic moduli of the material, for orthotropic material.

The strain displacement relations for small displacements are:

$$\varepsilon_x = \frac{\partial u}{\partial x} \quad (6)$$

$$\varepsilon_y = \frac{\partial v}{\partial y} \quad (7)$$

$$\gamma_{xy} = \frac{\partial v}{\partial x}+\frac{\partial u}{\partial y} \quad (8)$$

Eliminating $\sigma_x$ between equations (1) and (2) the following equations are obtained, which can be written in matrix form as;

$$\frac{\partial}{\partial y}\begin{bmatrix} u \\ v \\ Y \\ X \end{bmatrix} = \begin{bmatrix} 0 & -\alpha & 0 & 1/G \\ C_1\alpha & 0 & C_2 & 0 \\ 0 & 0 & 0 & -\alpha \\ C_3\alpha^2 G & 0 & C_1\alpha & 0 \end{bmatrix}\begin{bmatrix} u \\ v \\ Y \\ X \end{bmatrix} \quad (9)$$

Where,

$$X=\tau_{xy},\; Y=\sigma_y=C_{12}\varepsilon_x+C_{22}\varepsilon_y$$

$$C_1=\frac{-a_{12}}{a_{22}};C_2=\frac{1}{Ga_{22}};C_3=\frac{a_{12}}{a_{22}}-a_{11}$$

and

$$a_{11}=\frac{C_{11}}{G},a_{12}=\frac{C_{12}}{G},a_{22}=\frac{C_{22}}{G}$$

The equation (9) can be expressed as:

$$\frac{\partial}{\partial y}\{S\}=[D]\{S\} \quad (10)$$

The solution of equation (10) is

$$\{S\}=\left[e^{[D]y}\right]\{S_0\} \quad (11)$$

Where $\{S_0\}$ is the vector of initial functions, being the value of the state vector {S} on the initial plane.
If $u_0$, $v_0$, $Y_0$ and $X_0$ are values of u, v, Y and X respectively, on the initial plane, then

$$\{S_0\}=[u_0,\quad v_0,\quad Y_0,\quad X_0]^T \quad (12)$$

$$[L]=e^{[D]y} \quad (13)$$

Expending (13) in the form of a series

$$[L]=[I]+y[D]+\frac{y^2}{2!}[D]^2+\ldots\ldots \quad (14)$$

## 3 APPLICATION OF MIF

An isotropic beam of length l, depth, H and loaded with uniformly distributed load p in the y- direction. The bottom plane of the beam is taken as the initial plane. Due to loading at the top plane of the beam one has

$X_0 = Y_0 = 0$

On the plane, y = H, the conditions are

X = 0, Y = -p

Y= -p on y=H,

After simplification yields the governing partial differential equation:

$$(L_{Yu}.L_{Xv} - L_{Yv}.L_{Xu})\phi = -p \quad (15)$$

Initial functions are obtained by substituting the value of $\Phi$:

$$u_0 = L_{Xv}\phi \quad (16)$$

$$v_0 = -L_{Xu}\phi \quad (17)$$

From the value of initial functions the value of displacements and stresses are obtained.

$$u = L_{uu}.u_0 + L_{uv}.v_0$$
$$v = L_{vu}.u_0 + L_{vv}.v_0$$
$$Y = L_{Yu}.u_0 + L_{Yv}.v_0$$
$$X = L_{Xu}.u_0 + L_{Xv}.v_0 \quad (18)$$

## 4 NUMERICAL EXAMPLE

The following values of beam dimensions are chosen for the particular problem,

H =1000 mm and 2000 mm ,l = 4000 mm

H is total thickness of beam and l is effective span of beam

The following material properties are taken:

$E=2.10*10^5$ N/mm$^2$, μ = 0.30, $G = 0.10*10^5$ N/mm$^2$

E and G are the young's modulus and shear modulus of elasticity. μ is the Poisson's ratio.

The boundary conditions of the simply supported edges are:

X = Y = v = 0, at x = 0 and x = l

u and v are the displacements in x and y direction respectively.

The boundary conditions are exactly satisfied by the auxiliary function.

$\Phi = A_1 \sin(\pi x/l)$

A uniformly distributed load P = 25N/mm is assumed, on the top surface of the beam.

## 5 RESULTS AND DISCUSSION

The value of auxiliary function $\Phi$ is obtained from equation (15) using this value of auxiliary function, the values of initial functions $u_0$ and $v_0$ is obtained from equation (16) and (17).

The initial functions are operated upon by the transfer matrix successively across each layer until theentire beam is analysed and the stresses at the top surface are again obtained. Governing equation (15) of desired order according to the requirements of a beam problem is obtained using MIF.

The values of $u_0$ and $v_0$ are substituted in expression (18) for obtaining the values of stresses and displacements. Results have been given in Table 1 and 2. The distribution of stresses and displacements across the depth of a simply supported beam for uniformly distributed loaded isotropic beam are shown in figure 1 to 10.

Table 1 Values of displacements and stresses at different depths for beam of 1000mm depth.

| H (mm) | u (N/mm$^2$) | v (N/mm$^2$) | Y (N/mm$^2$) | X (N/mm$^2$) | $\sigma_x$ (N/mm$^2$) |
|---|---|---|---|---|---|
| 0 | 30.3 | 93.1 | 0.00 | 0.00 | -524.3 |
| 100 | 23.2 | 93.3 | 1.16 | 36.3 | -400.9 |
| 200 | 16.3 | 93.5 | 4.26 | 63.1 | -282.2 |
| 300 | 9.73 | 93.7 | 8.67 | 81.1 | -166.7 |
| 400 | 3.21 | 93.8 | 13.7 | 90.7 | -52.72 |
| 500 | -3.30 | 94.0 | 18.9 | 92.3 | 61.42 |
| 600 | -9.89 | 94.2 | 23.6 | 86.6 | 177.7 |
| 700 | -16.6 | 94.4 | 27.2 | 74.0 | 296.7 |
| 800 | -23.7 | 94.7 | 29.0 | 55.0 | 421.3 |
| 900 | -31.1 | 95.1 | 28.4 | 30.2 | 552.5 |
| 1000 | -38.9 | 95.6 | 24.9 | 0.00 | 692.3 |

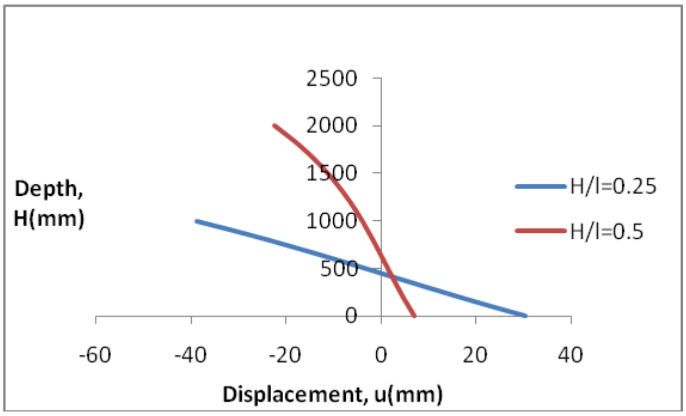


Figure 1. Variation of "u" through the thickness of beam for H/l= 0.25and H/l=0.5.

Table 2 Values of displacements and stresses at different depths for beam of 2000mm depth.

| H (mm) | u ($N/mm^2$) | v ($N/mm^2$) | Y ($N/mm^2$) | X ($N/mm^2$) | $\sigma_x$ ($N/mm^2$) |
|---|---|---|---|---|---|
| 0 | 7.01 | 16.2 | 0.00 | 0.00 | -121.2 |
| 200 | 4.62 | 16.3 | 1.03 | 15.7 | -79.67 |
| 400 | 2.45 | 16.4 | 3.55 | 25.4 | -41.75 |
| 600 | 0.38 | 16.5 | 6.72 | 30.1 | -5.182 |
| 800 | -1.71 | 16.7 | 9.66 | 30.6 | 32.34 |
| 1000 | -3.98 | 17.0 | 11.5 | 28.0 | 73.13 |
| 1200 | -6.55 | 17.3 | 11.4 | 23.1 | 119.5 |
| 1400 | -9.55 | 17.9 | 8.60 | 16.9 | 173.7 |
| 1600 | -13.1 | 18.6 | 2.10 | 10.4 | 238.2 |
| 1800 | -17.3 | 19.6 | -8.89 | 4.45 | 315.2 |
| 2000 | -22.4 | 20.8 | -25.1 | 0.00 | 407.1 |

It decreases with increase in the depth span ratio.For H/l= 0.25 the variation of 'u' is almost linear but for H/l=0.50 its variation is more at the top surface of beam.

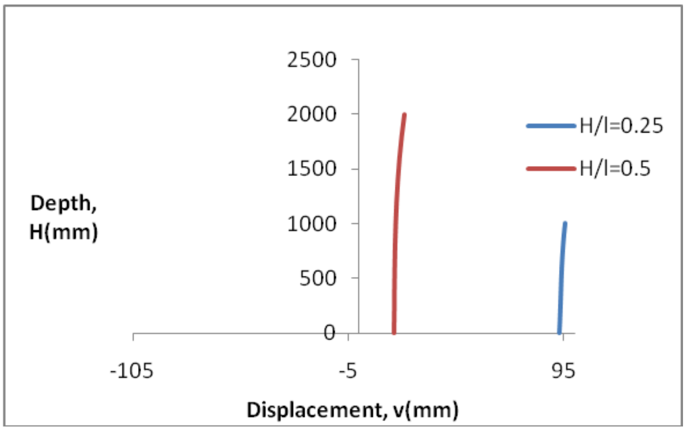


Figure 2. Variation of "v" through the thickness of beam for H/l= 0.25 and H/l=0.5.

It decreases with increase in the depth span ratio.The displacement 'v' is uniform throughout the depth for H/l= 0.25 and there is small variation for H/l=0.50.

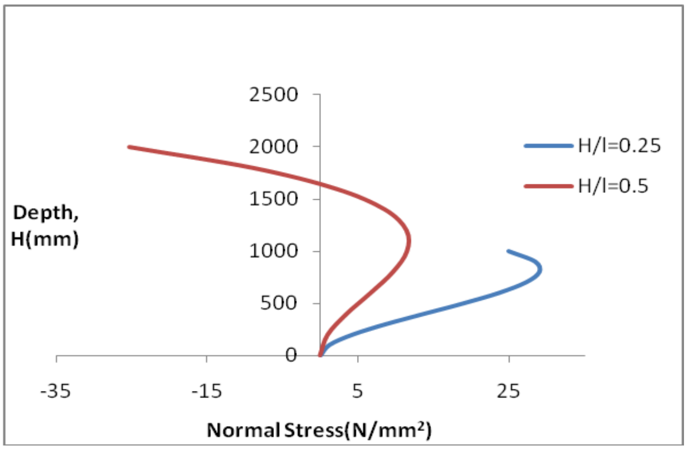


Figure 3. Variation of "Normal stress (Y)" through the thickness of beam for H/l= 0.25 and H/l=0.5.

Variation of normal stress across the depth increases with increase in depth span ratio. It is equal to intensity of loading at the top surface of beam in both the cases.

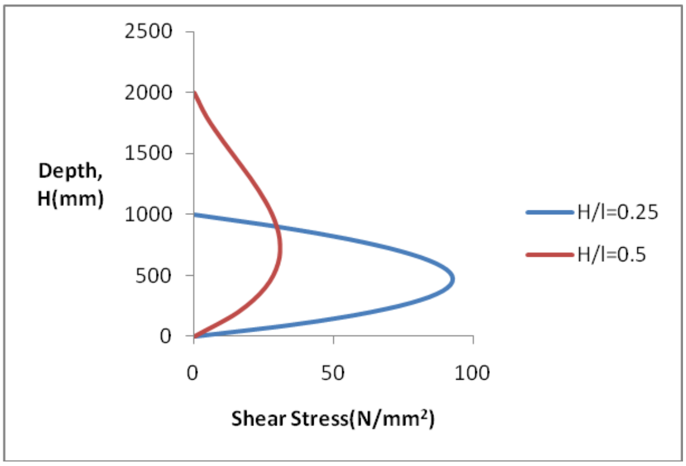


Figure 4. Variation of "Shear stress (X)" through the thickness of beam for H/l= 0.25and H/l=0.5.

The shear stress is maximum at mid depth in case of H/l= 0.25 and it is just below the mid depth in case of H/l=0.50. Shear stress decreases with the increase in depth span ratio.

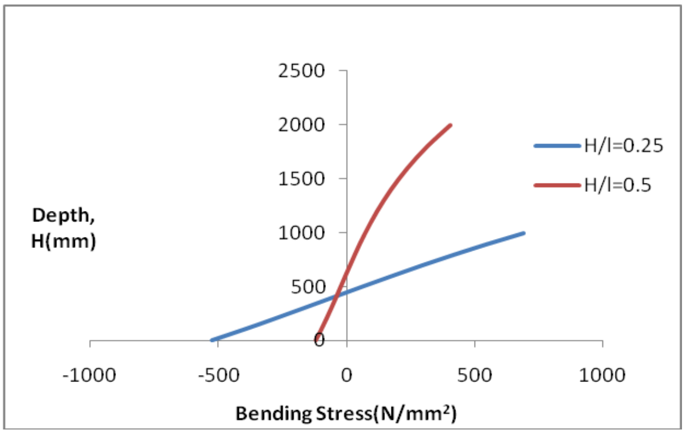


Figure 5. Variation of "Bending tress ($\sigma_x$)" through the thickness of beam for H/l= 0.25 and H/l=0.5.

The variation of bending stress is almost linear in case of H/l= 0.25 but in the case of H/l=0.50 it is not linear near the top surface of the beam. The neutral axis shifts from mid depth towards bottom surface. It shows that the deep beam action comes into effect even when H/l= 0.25.

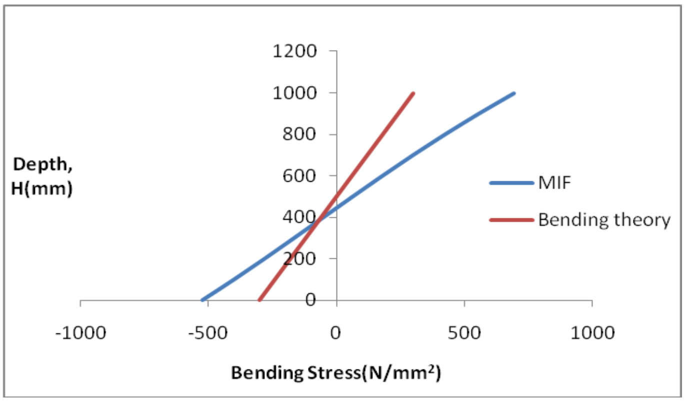


Figure 6. Comparison of MIF results with bending theory for "Bending stress ($\sigma_x$)" through the thickness of beam for H/l=0.25.

Results obtained with MIF are compared with the results obtained by bending theory and it is observed that in case of bending theory neutral axis lie in the middle of the beam section but in case of MIF neutral axis is below the mid depth.It shows that the deep beam action comes into effect even when H/l= 0.25. This is because no assumption regarding the position of neutral axis is taken in MIF .It incorporates the actual position of neutral axis by itself.

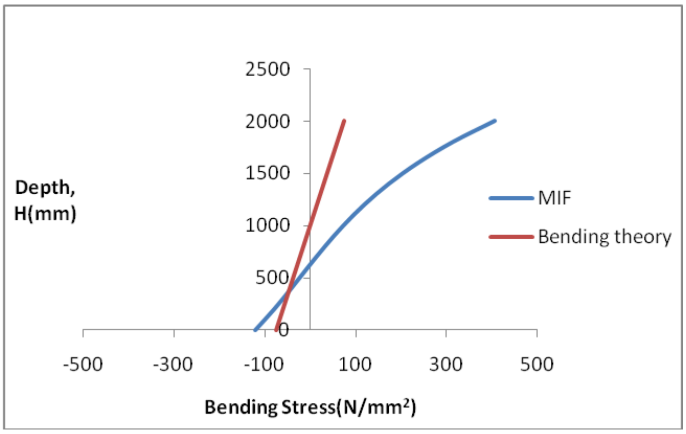


Figure 7. Comparison of MIF results with bending theory for "Bending stress ($\sigma_x$)" through the thickness of beam for H/l=0.50.

Warping of section is shown by MIF as deep beam action is clearly seen. But in case of bending theory variation of stress is uniform.MIF shows that the neutral axis shifts from mid depth towards bottom surface. For small depth both the theories gives almost same results.

## 6 CONCLUSIONS

MIF is different from other methods because it does not take assumptions regarding the physical behaviour of beams. Also no shear correction factor is required in this method. This method also incorporates the position of neutral axis by itself.For small depth MIF and bending theory gives almost same results. But as the depth increases the results are vaying.

The normal stress equal to the intensity of loading and shear stress equal to zero at the top of beam are obtain, this shows that MIF is successfully applied for the analysis of isotropic beams. The nature of the curves obtained for stresses and displacements is similar to those obtained by other theories.

Deep beam action is clearly seen at H/l= 0.25 and 0.50. MIF yields correct results for both shallow and deep beam. In the theories based on assumptions this effect is not seen. Hence it can be successfully used as an alternative approach for the analysis of beams. It gives accurate results in case of small thickness, large thickness and layered members or composite members.

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