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Estimation of Membership Function of Design Variables Using HDMR and FFT

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ABSTRACT: This paper presents an inverse reliability analysis to determine the unknown design parameters such that prescribed reliability indices are attained in the presence of mixed uncertain variables. The proposed computational procedure involves the failure probability estimation using High Dimensional Model Representation, transformation technique to obtain the contribution of the fuzzy variables to the convolution integral, convolution using fast Fourier transform, and update of reliability index and most probable point. This is a versatile method that can solve even highly nonlinear problems or the problems with multiple parameters. The methodology developed is applicable for inverse reliability analysis involving any number of fuzzy variables and random variables with any kind of distribution. The accuracy and efficiency of the proposed method is demonstrated through three examples involving explicit/implicit performance functions.

Keywords: Inverse reliability analysis, High Dimensional Model Representation, Random variables, Fuzzy variables; Fast Fourier transform; Convolution integral

1 INTRODUCTION

The solution of inverse reliability problems aims at determining the unknown design parameters such that the prescribed reliability indices are attained. One way to solve the inverse reliability problems is through trial and error procedure, using a forward reliability method like first-order reliability method (FORM) and varying the design parameters until the achieved reliability index matches the required target (Li and Foschi, 1998). However, the trial and error procedures are inefficient and involve difficulties resulting from repetitive forward reliability analysis. Even though the forward reliability analysis using SORM may be accurate (in which the second-order sensitivities of the function are required), the SORM-based inverse reliability analysis is rather difficult to develop (Lee et al., 2008).

Lee and Kwak (1987) developed an inverse reliability formulation for low probability failure design. Winterstein et al. (1993) developed an inverse first-order reliability method (inverse FORM) and utilized this method for the estimation of design loads associated with specified target reliability levels in offshore structures. An extension of the method was developed by Der Kiureghian et al. (1994) for general limit state functions. To overcome the drawbacks of the inverse FORM, Cheng et al. (2007) proposed an artificial neural network (ANN)-based inverse FORM. Cheng and Li (2009) adopted a polynomial-based response surface method to improve the accuracy than ANN-based inverse FORM. In most of the works cited above, the inverse reliability problem is either solved by using sampling techniques or by using inverse-FORM. As an alternative, Lee et al. (2008) proposed an inverse reliability analysis method for reliability-based design optimization (RBDO) of nonlinear and multi-dimensional systems by developing the most probable point (MPP)-based dimension reduction method (DRM).

Traditionally, inverse reliability methods require complete statistical information of uncertainties. These uncertainties are treated stochastically and assumed to follow certain probability distributions. However, in many practical engineering applications, the distributions of some random variables may not be precisely known or uncertainties may not be appropriately represented with probability distributions. Consequently, an alternative category, namely the non-probabilistic approach (Ben-Haim, 1995), has been rapid developed for describing uncertainty with incomplete statistical information by a fuzzy set or a convex set. In the fuzzy set method (Möller and Beer, 2004), the fuzzy failure probability of structures is assessed based on membership function representation of the observed/ measured inputs. However all the methods discussed above consider either random variables or fuzzy input, but do not accommodate a combination of both types of variables. Hence there is considerable interest in developing efficient methods for dealing with problems comprising of mixed uncertain variables.

In the design problem with both statistical random variables and fuzzy variables, if the random variables are converted into fuzzy variables by generating membership functions, the method may yield a design that is too conservative because treating the random variables as fuzzy variables loses accuracy of the uncertainties. On the other hand, treating fuzzy variables as random variables by adopting approximate probability distributions may lead to an unreliable optimum design.

Therefore, in this paper a novel solution procedure for inverse reliability problems with implicit response functions without requiring the derivatives of the response functions with respect to the uncertain variables, is proposed to determine the unknown design parameters such that prescribed reliability indices are attained in the presence of mixed uncertain (both random and fuzzy) variables.

2 CONCEPT OF HIGH DIMENSIONAL MODEL REPRESENTATION

High Dimensional Model Representation (HDMR) is a general set of quantitative model assessment and analysis tools for capturing the high-dimensional relationships between sets of input and output model variables (Chowdhury and Rao, 2008). It approximates multivariate functions in such a way that the component functions of the approximation are ordered starting from a constant and gradually approaching to multi-variance as we proceed along the terms like first-order, second-order and so on. Practically for most well-defined physical systems, only relatively low order correlations of the input variables are expected to have a significant effect on the overall response. HDMR expansion utilizes this property to present an accurate hierarchical representation of the physical system (Rabitz and Alis, 1999; Rao and Chowdhury, 2008; Sobol, 2003).

Degree of accuracy of reliability estimation depends on the accurate representation of the limit state/performance function. Computational complexity for the generation of response surface of implicit limit state/performance function arises due to increase in number of input variables, while using conventional response surface in conjunction with design of experiments. The concept of HDMR expansions is introduced here for the purpose of approximating the limit state/performance function most accurately and efficiently, when the number of input variables is large.

Let the N–dimensional vector $\boldsymbol{x} = \{x_1, x_2, \ldots, x_N\}$ represent the input variables of the model under consideration, and the response function as $g(\boldsymbol{x})$. Since the influence of the input variables on the response function can be independent and/or cooperative, HDMR expresses the response $g(\boldsymbol{x})$ as a hierarchical correlated function expansion in terms of the input variables as

$$\begin{aligned} g(\boldsymbol{x}) = g_0 + \sum_{i=1}^{N} g_i(x_i) + \sum_{1\le i_1 < i_2 \le N} g_{i_1 i_2}\left(x_{i_1}, x_{i_2}\right) + \ldots \\ + \sum_{\substack{1\le i_1<\ldots \\ <i_l \le N}} g_{i_1 i_2 \ldots i_l}\left(x_{i_1}, x_{i_2}, \ldots, x_{i_l}\right) + \ldots + g_{12\ldots N}\left(x_1, x_2, \ldots, x_N\right) \end{aligned} , \quad (1)$$

Where, g_0 is a constant term representing the zeroth-order component function or the mean response of $g(\boldsymbol{x})$. The function $g_i(x_i)$ is a first-order term expressing the effect of variable x_i acting alone, although generally nonlinearly, upon the output $g(\boldsymbol{x})$. The function $g_{i_1 i_2}\left(x_{i_1}, x_{i_2}\right)$ is a second-order term which describes the cooperative effects of the variables x_{i_1} and x_{i_2} upon the output $g(\boldsymbol{x})$. The higher order terms give the cooperative effects of increasing numbers of input variables acting together to influence the output $g(\boldsymbol{x})$. The last term $g_{12,\cdots,N}\left(x_1, x_2, \cdots, x_N\right)$ contains any residual dependence of all the input variables locked together in a cooperative way to influence the output $g(\boldsymbol{x})$.

Once all the relevant component functions in Equation 1 are determined and suitably represented, then the component functions constitute HDMR, thereby replacing the original computationally expensive method of calculating $g(\boldsymbol{x})$ by the computationally efficient model. Usually the higher order terms in Equation 1 are negligible such that HDMR with only low order correlations to second-order, amongst the input variables are typically adequate in describing the output behavior. This has been verified in a number of computational studies where HDMR expansions up to second-order are often sufficient to describe the outputs of many realistic systems. Therefore it is expected that HDMR expansion converges very rapidly.

With cut-HDMR method, first a reference point $\boldsymbol{c} = \{c_1, c_2, \cdots, c_N\}$ is defined in the variable space. In the convergence limit, cut-HDMR is invariant to the choice of reference point $\boldsymbol{c}$. In practice, $\boldsymbol{c}$ is chosen within the neighborhood of interest in the input space. The expansion functions are determined by evaluating the input-output responses of the system relative to the defined reference point $\boldsymbol{c}$ along associated lines, surfaces, sub-volumes, etc. (i.e. cuts) in the input variable space. This process reduces to the following relationship for the component functions in Equation 1,

$$g_0 = g(\boldsymbol{c}), \tag{2}$$

$$g_i(x_i) = g(x_i, \boldsymbol{c}^i) - g_0, \tag{3}$$

$$g_{i_1 i_2}(x_{i_1}, x_{i_2}) = g(x_{i_1}, x_{i_2}, \boldsymbol{c}^{i_1 i_2}) - g_{i_1}(x_{i_1}) - g_{i_2}(x_{i_2}) - g_0, \tag{4}$$

Where, $g(x_i, \boldsymbol{c}^i) = g(c_1, c_2, \ldots, c_{i-1}, x_i, c_{i+1}, \ldots, c_N)$ denotes that all the input variables are at their reference point values except x_i. The g_0 term is the output response of the system evaluated at the reference point $\boldsymbol{c}$. The higher order terms are evaluated as cuts in the input variable space through the reference point. Therefore, each first-order term $g_i(x_i)$ is evaluated along its variable axis through the reference point. Each second-order term $g_{i_1 i_2}(x_{i_1}, x_{i_2})$ is evaluated in a plane defined by the binary set of input variables x_{i_1}, x_{i_2} through the reference point, etc. The process of subtracting off the lower order expansion functions removes their dependence to assure a unique contribution from the new expansion function. Considering terms up to first-order in Equation 1 yields,

$$g(\boldsymbol{x}) = g_0 + \sum_{i=1}^{N} g_i(x_i) + \mathrm{R}_2. \tag{5}$$

Substituting Equations 2 and 3 into Equation 5 leads to;

$$g(\boldsymbol{x}) = \sum_{i=1}^{N} g(c_1, \ldots, c_{i-1}, x_i, c_{i+1}, \ldots, c_N) - (N-1)g(\boldsymbol{c}) + \mathrm{R}_2. \tag{6}$$

Now consider first-order approximation of $g(\boldsymbol{x})$,

$$\tilde{g}(\boldsymbol{x}) \equiv g(x_1, x_2, \ldots, x_N) = \sum_{i=1}^{N} g(c_1, \ldots, c_{i-1}, x_i, c_{i+1}, \ldots, c_N) - (N-1)g(\boldsymbol{c}) \tag{7}$$

Comparison of Equations 6 and 7 indicates that the first-order approximation leads to the residual error $g(\boldsymbol{x}) - \tilde{g}(\boldsymbol{x}) = \mathrm{R}_2$, which includes contributions from terms of two and higher order component functions. The notion of 0^{th}, 1^{st}, etc. in HDMR expansion should not be confused with the terminology used either in the Taylor series or in the conventional least-squares based regression model. It can be shown that, the first order component function $g_i(x_i)$ is the sum of all the Taylor series terms which contain and only contain variable x_i. Hence first-order HDMR approximations should not be viewed as first-order Taylor series expansions nor do they limit the nonlinearity of $g(\boldsymbol{x})$.

Furthermore, the approximations contain contributions from all input variables. Thus, the infinite number of terms in the Taylor series is partitioned into finite different groups and each group corresponds to one cut-HDMR component function. Therefore, any truncated cut-HDMR expansion provides a better approximation and convergent solution of $g(\boldsymbol{x})$ than any truncated Taylor series because the latter only contains a finite number of terms of Taylor series. Furthermore, the coefficients associated with higher dimensional terms are usually much smaller than that with one-dimensional terms. As such, the impact of higher dimensional terms on the function is less, and therefore, can be neglected. Compared with the FORM and SORM which retain only linear and quadratic terms, respectively, first-order HDMR approximation $\tilde{g}(\boldsymbol{x})$ provides more accurate representation of the original implicit limit state/performance function $g(\boldsymbol{x})$.

3 INVERSE STRUCTURAL RELIABILITY ANALYSIS USING HDMR AND FFT

The objective of the inverse reliability analysis using HDMR and FFT is to find a new MPP, denoted by $\boldsymbol{x}^*_{\mathrm{HDMR}}$, which will be then used in the subsequent iteration of analysis. The proposed computational procedure involves the following three steps: estimation of failure probability in presence of mixed uncertain variables, reliability index update, and MPP update.

Let the *N*–dimensional input variables vector $\boldsymbol{x} = \{x_1, x_2, \ldots, x_N\}$, which comprises of r number of random variables and f number of fuzzy variables be divided as, $\boldsymbol{x} = \{x_1, x_2, \ldots, x_r, x_{r+1}, x_{r+2}, \ldots, x_{r+f}\}$ where the sub-vectors $\{x_1, x_2, \ldots, x_r\}$ and $\{x_{r+1}, x_{r+2}, \ldots, x_{r+f}\}$ respectively group the random variables and the fuzzy variables, with $N = r + f$. Then the first-order approximation of $g(\boldsymbol{x})$ in Equation 7 can be divided into three parts, the first part with only the random variables, the second part with only the fuzzy variables and the third part is a constant which is the output response of the system evaluated at the reference point $\boldsymbol{c}$, as follows.

$$\tilde{g}(\boldsymbol{x}) = \sum_{i=1}^{r} g\left(x_i, \boldsymbol{c}^i\right) + \sum_{i=r+1}^{N} g\left(x_i, \boldsymbol{c}^i\right) - (N-1)\, g(\boldsymbol{c}) \tag{8}$$

The joint membership function of the fuzzy variables part is obtained using suitable transformation of the variables $\{x_{r+1}, x_{r+2}, \ldots, x_N\}$ and interval arithmetic algorithm. Using this approach, the minimum and maximum values of the fuzzy variables part are obtained at each α-cut. Using the bounds of the fuzzy variables part at each α-cut along with the constant part and the random variables part in Equation 8, the joint density functions are obtained by performing the convolution using FFT in the rotated Gaussian space at the MPP, which upon integration yields the bounds of the failure probability.

3.1 *Transformation of Fuzzy Variables*

Optimization techniques are required to obtain the minimum and maximum values of a nonlinear response within the bounds of the interval variables. This procedure is computationally expensive for problems with implicit limit state functions, as optimization requires the function value and gradient information at several points in the iterative process. But, if the function is expressed as a linear combination of interval variables, then the bounds of the response can be expressed as the summation of the bounds of the individual variables. Therefore, fuzzy variables part of the nonlinear limit state function in Equation 8 is expressed as a linear combination of intervening variables by the use of first-order HDMR approximation in order to apply an interval arithmetic algorithm, as follows

$$\sum_{i=r+1}^{N} g\left(x_i, \boldsymbol{c}^i\right) = z_1 + z_2 + \ldots + z_f , \tag{9}$$

where, $z_i = \left(\beta_i x_i + \gamma_i\right)^{\kappa}$ is the relation between the intervening and the original variables with κ being order of approximation taking values $\kappa = 1$ for linear approximation, $\kappa = 2$ for quadratic approximation, $\kappa = 3$ for cubic approximation, and so no. The bounds of the intervening variables can be determined using transformations. If the membership functions of the intervening variables are available, then at each α-cut, interval arithmetic techniques can be used to estimate the response bounds at that level. Similar transformation techniques for membership functions to obtain minimum and maximum values of a nonlinear response at each α-cut are adopted by Adduri and Penmetsa (2008), except that a second-order response surface model is used for the original response approximation, however the proposed first-order HDMR provides better approximation $\tilde{g}(\boldsymbol{x})$ of the original limit state function $g(\boldsymbol{x})$.

The use of transformation techniques facilitates the determination of the minimum and maximum values of the fuzzy variables that correspond to the extreme values of the response at a particular level without the use of optimization techniques in addition to fixing the dependency issues associated with nonlinear functions. Moreover, this is an analytical procedure in which there is little room for errors.

3.2 *Estimation of Failure Probability using FFT*

Concept of FFT can be applied to the problem if the limit state function is in the form of a linear combination of independent variables and when either the marginal density or the characteristic function of each basic random variable is known. Even if the function of the basic variables is nonlinear, an appropriate transformation of the basic random variables could yield a linear function of independent random variables. To achieve this linear function, the limit state function can be approximated by using a first-order Taylor series expansion, but this gives very poor accuracy. In the present study HDMR concepts are used to express the random variables part along with the values of the constant part and the fuzzy variables part at each α-cut, which depends on $\{x_1, x_2, \ldots, x_r\} \in \Re^r$, as a linear combination of lower order component functions. The steps involved in the proposed method for failure probability estimation as follows:

(i) If $\boldsymbol{u} = \{u_1, u_2, \ldots, u_r\}^T \in \Re^r$ is the standard Gaussian variable, let $\boldsymbol{u}^* = \{u_1^*, u_2^*, \ldots, u_r^*\}^T$ be the MPP or design point, determined by a standard nonlinear constrained optimization. The MPP has a distance β_{HL}, which is commonly referred to as the Hasofer–Lind reliability index. Construct an orthogonal matrix $\boldsymbol{R} \in \Re^{r \times r}$ whose r–th column is $\boldsymbol{\alpha}^* = \boldsymbol{u}^* / \beta_{HL}$, i.e., $\boldsymbol{R} = [\boldsymbol{R}_1 \,|\, \boldsymbol{\alpha}^*]$ where $\boldsymbol{R}_1 \in \Re^{r \times r-1}$ satisfies $\boldsymbol{\alpha}^{*T} \boldsymbol{R}_1 = 0 \in \Re^{1 \times r-1}$. The matrix $\boldsymbol{R}$ can be obtained, for example, by Gram–Schmidt orthogonalization. For an orthogonal transformation $\boldsymbol{u} = \boldsymbol{R}\boldsymbol{v}$. Let $\boldsymbol{v} = \{v_1, v_2, \ldots, v_r\}^T \in \Re^r$ be the rotated Gaussian space with the associated

MPP $\boldsymbol{v}^* = \left\{v_1^*, v_2^*, \ldots, v_r^*\right\}^T$. Note that in the rotated Gaussian space the MPP is $\boldsymbol{v}^* = \{0, 0, \ldots, \beta_{HL}\}^T$. The transformed limit state function $g(\boldsymbol{v})$ therefore maps the random variables along with the values of the constant part and the fuzzy variables part at each α-cut, into rotated Gaussian space $\boldsymbol{v}$. First-order HDMR approximation of $g(\boldsymbol{v})$ in rotated Gaussian space $\boldsymbol{v}$ with $\boldsymbol{v}^* = \left\{v_1^*, v_2^*, \ldots, v_r^*\right\}^T$ as reference point can be represented as follows:

$$\begin{aligned}\tilde{g}(\boldsymbol{v}) &\equiv g(v_1, v_2, \ldots, v_r) \\ &= \sum_{i=1}^{r} g\left(v_1^*, \ldots, v_{i-1}^*, v_i, v_{i+1}^*, \ldots, v_r^*\right). \\ &\quad -(r-1)\, g\left(\boldsymbol{v}^*\right)\end{aligned} \tag{10}$$

(ii) In addition to the MPP as the chosen reference point, the accuracy of first-order HDMR approximation in Equation 10 may depend on the orientation of the first $r-1$ axes. In the present work, the orientation is defined by the matrix $\boldsymbol{R}$.

(iii) Since the terms $g\left(v_1^*, \ldots, v_{i-1}^*, v_i, v_{i+1}^*, \ldots, v_r^*\right)$ are the individual component functions and are independent of each other, Equation 10 can be rewritten as,

$$\tilde{g}(\boldsymbol{v}) = a + \sum_{i=1}^{r} g\left(v_i, \boldsymbol{v}^{*i}\right), \tag{11}$$

where $a = -(r-1)\, g\left(\boldsymbol{v}^*\right)$.

(iv) New intermediate variables are defined as

$$y_i = g\left(v_i, \boldsymbol{v}^{*i}\right). \tag{12}$$

(v) The purpose of these new variables is to transform the approximate function into the following form

$$\tilde{g}(\boldsymbol{v}) = a + y_1 + y_2 + \cdots + y_r. \tag{13}$$

(vi) Due to rotational transformation in $\boldsymbol{v}$-space, component functions y_i in Equation 10 are expected to be linear or weakly nonlinear function of random variables v_i. In this work both linear and quadratic approximations of y_i are considered.

(vii) Consider $y_i = b_i + c_i\, v_i$ for linear and $y_i = b_i + c_i\, v_i + e_i\, v_i^2$ for quadratic approximations. The coefficients $b_i \in \Re$, $c_i \in \Re$ and $e_i \in \Re$ (non-zero) are obtained by least-squares approximation from exact or numerically simulated conditional responses $\left\{g\left(v_i^1, \boldsymbol{v}^{*i}\right), g\left(v_i^2, \boldsymbol{v}^{*i}\right), \cdots, g\left(v_i^n, \boldsymbol{v}^{*i}\right)\right\}^T$ at n sample points along the variable axis v_i. Then Equation 13 results in respectively for linear and quadratic approximations as

$$\tilde{g}(\boldsymbol{v}) \equiv a + \sum_{i=1}^{r} \left(b_i + c_i\, v_i\right), \tag{14}$$

and

$$\tilde{g}(\boldsymbol{v}) \equiv a + \sum_{i=1}^{r} \left(b_i + c_i\, v_i + e_i\, v_i^2\right). \tag{15}$$

(viii) Since v_i follows standard Gaussian distribution, marginal density of the intermediate variables y_i can be easily obtained by simple transformation (using chain rule).

$$p_{Y_i}(y_i) = p_{V_i}(v_i) \left|\frac{dv_i}{dy_i}\right|. \tag{16}$$

(ix) Now the approximation is a linear combination of the intermediate variables y_i. Therefore, the joint density of $\tilde{g}(\boldsymbol{v})$, which is the convolution of the individual marginal density of the intervening variables y_i, can be expressed as follows:

$$p_{\tilde{G}}(\tilde{g}) = p_{Y_1}(y_1) * p_{Y_2}(y_2) * \ldots * p_{Y_r}(y_r), \tag{17}$$

where $p_{\tilde{G}}(\tilde{g})$ represents joint density of the transformed limit state function $\tilde{g}(\boldsymbol{v})$.

(x) Applying FFT on both sides of Equation 17, leads to

$$\begin{aligned}FFT\left[p_{\tilde{G}}(\tilde{g})\right] &= FFT\left[p_{Y_1}(y_1)\right] \\ &* FFT\left[p_{Y_2}(y_2)\right] * \ldots * FFT\left[p_{Y_r}(y_r)\right]\end{aligned}. \tag{18}$$

(xi) By applying inverse FFT on both side of Equation 18, joint density of the limit state function $\tilde{g}(\boldsymbol{v})$ is obtained.

(xii) The probability of failure is given by the following equation

$$P_F^{\text{HDMR}} = \int_{-\infty}^{0} p_{\tilde{G}}(\tilde{g})\, d\tilde{g}. \tag{19}$$

(xiii) After computing the probability of failure P_F^{HDMR} using coupled HDMR-FFT technique, the corresponding reliability index β_{HDMR} can be obtained by

$$\beta_{\text{HDMR}} = -\Phi^{-1}\left(P_F^{\text{HDMR}}\right), \tag{20}$$

where $\Phi(\bullet)$ is the cumulative distribution function of a standard Gaussian random variable.

3.3 *Reliability Index Update Procedure*

As expected it is very likely that the β_{HDMR} is not the same as the target reliability index $\beta_t = -\Phi^{-1}\left(P_F^{\text{Tar}}\right)$, and hence, using the difference between these two reliability indices, a recursive formula is obtained as

$$\beta^{(k+1)} \cong \beta^{(k)} - \left(\beta_{\text{HDMR}} - \beta_t\right), \tag{21}$$

where $\beta^{(k)}$ is the reliability index at the current step, with $\beta^{(0)} = \beta_t$ at the initial step.

3.4 *MPP Update Procedure*

The updated MPP is approximated as

$$\boldsymbol{u}_{k+1}^* \cong \frac{\beta^{(k+1)}}{\beta^{(k)}} \boldsymbol{u}_k^* \quad \text{or} \quad \boldsymbol{v}_{k+1}^* \cong \frac{\beta^{(k+1)}}{\beta^{(k)}} \boldsymbol{v}_k^*, \tag{22}$$

The updated MPP obtained through Equation 22 is called the coupled HDMR-FFT based MPP, denoted by $\boldsymbol{u}_{\text{HDMR}}^*$ in $\boldsymbol{U}$-space or $\boldsymbol{x}_{\text{HDMR}}^*$ in $\boldsymbol{X}$-space.

3.5 *Detailed Algorithm of Proposed Computational Procedure*

The various steps involved in the proposed computational procedure for inverse reliability problems with implicit response functions in the presence of mixed uncertain (both random and fuzzy) variables is as follows:

(i) Find MPP in the rotated Gaussian space using a given reliability index $\beta^{(k)}$.

(ii) Calculate the probability of failure P_F^{HDMR} and the corresponding reliability index β_{HDMR} using coupled HDMR-FFT technique.

(iii) Using Equations 21 and 22 respectively, update the reliability index from $\beta^{(k)}$ to $\beta^{(k+1)}$ and MPP from $\boldsymbol{u}_k^*$ ($\boldsymbol{v}_k^*$) to $\boldsymbol{u}_{k+1}^*$ ($\boldsymbol{v}_{k+1}^*$).

(iv) Find a new coupled HDMR-FFT based MPP $\boldsymbol{x}_{\text{HDMR}}^*$.

(v) Compare $\boldsymbol{x}_{\text{HDMR}}^*$ and $\boldsymbol{x}^*$.

(vi) Repeat the above steps until converged.

(vii) Using the minimum and maximum values of the fuzzy variables part (Equation 8) at each α-cut, the bounds of the design variables can be obtained by adopting the above procedure.

4 NUMERICAL EXAMPLES

Three numerical examples involving explicit hypothetical mathematical functions and implicit functions from structural mechanics problems are presented to illustrate the performance of the proposed inverse reliability method. In the present work, transformation of fuzzy variables and FFT are conducted in conjunction with HDMR based approximation. To obtain the approximation of the HDMR component functions of fuzzy variables part of the nonlinear limit state function in Equation 8, n sample points x_{iL}, $x_{iM} - (n-3)\left(x_{iM} - x_{iL}\right)/(n-1)$, $x_{iM} - (n-5)\left(x_{iM} - x_{iL}\right)/(n-1)$, …, x_{iM}, …, $x_{iM} + (n-5)(x_{iU} - x_{iM})/(n-1)$, $x_{iM} + (n-3)(x_{iU} - x_{iM})/(n-1)$, x_{iU} are deployed along axis of each of the fuzzy variable x_i having triangular membership function with the triplet number $[x_{iL}, x_{iM}, x_{iU}]$.

Similarly to obtain the HDMR component functions of random variables part of the nonlinear function, n sample points $\mu_i - (n-1)\sigma_i/2$, $\mu_i - (n-3)\sigma_i/2$, …, μ_i, …, $\mu_i + (n-3)\sigma_i/2$, $\mu_i + (n-1)\sigma_i/2$ are deployed along axis of each of the random variable x_i with mean μ_i and standard deviation σ_i. Sampling schemes for HDMR approximation of a function having one variable (x) and two variables (x_1 and x_2) are shown in Figures 1(a) and 1(b) respectively.

If N and n respectively denote the number of uncertain variables, the number of sample points taken along each of the variable axis, then using first-order HDMR approximation the total cost of original function evaluation entails a maximum of $N \times (n-1)+1$ by the proposed method. The efficiency and robustness of the proposed method is expected to increase with increase in the complexity of the structure, number of uncertain variables.

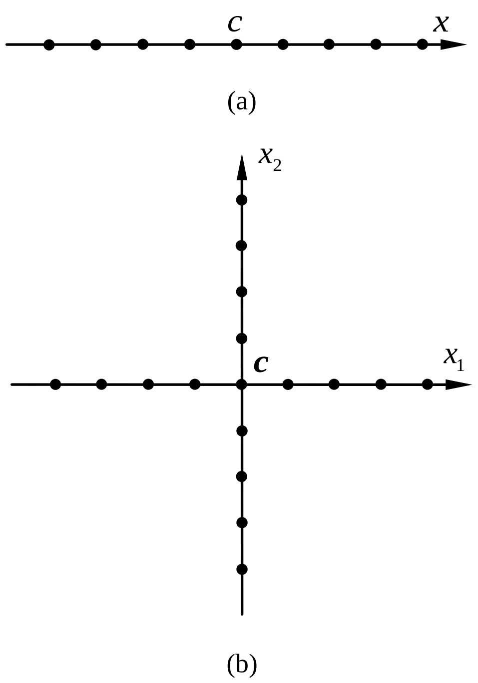

Figure 1. Sampling scheme for first-order HDMR:
(a) For a function having one variable (x); and
(b) For a function having two variables (x_1 and x_2)

4.1 *Hypothetical Limit State Function with Two Variables*

This example considers a hypothetical limit state function with two random variables of the following form, studied earlier by Du et al (2004):

$$g(\boldsymbol{x}) = x_2 + (x_1+0.25)^2 - (x_1+0.25)^3 - (x_1+0.25)^4 - 4, \quad (23)$$

where $x_1 = N(0.0,1.0)$ and $x_2 = N(0.0,1.0)$; $N(\mu,\sigma)$ stands for a normal distribution with mean μ and standard deviation σ. Here our interest is to find the design variables (MPP) x_1^* and x_2^*, such that the target reliability index $\beta_t = 3.0$ (which corresponds to a failure probability $P_F = 0.0013$) is achieved.

The limit state function given in Equation 23 is approximated using first-order HDMR by deploying n sample points along each of the variable axis, and taking $\boldsymbol{c} = (0,0)$ as reference point. Using the proposed inverse reliability method in conjunction with linear and quadratic approximations the effect of number of sample points is studied by varying n from 3 to 9. Table 1 presents the comparison of the function $g(\boldsymbol{x})$ value, x_1^* and x_2^* values at the target reliability index, obtained using the proposed method with the values reported by Du et al. (2004) and the Sequential Quadratic Programming (SQP).

Table 1 also presents the computational effort in the number of function evaluations required for each method. It can be observed that $n=7$ in conjunction with quadratic approximation provides the optimum number of function calls with maximum accuracy in obtaining the design variables. Table 1 clearly demonstrates the computational efficiency of the proposed methodology.

Table 1. Comparison of the function values

Method	$\boldsymbol{x}^*$	$g(\boldsymbol{x}^*)$	No. of fn. evaluations
Du et. al.(2004)	(–1.3503, 2.6789)	–0.2440	20
SQP	(–1.5270, 2.9962)	–0.9332	28
Lin. Apprx. ($n=3$)	(–1.2569, 2.7551)	–0.2381	5
Quad. Apprx. ($n=3$)	(–1.2345, 2.7793)	–0.2366	5
Lin. Apprx. ($n=5$)	(–1.2963, 2.7349)	–0.2234	9
Quad. Apprx. ($n=5$)	(–1.2548, 2.7724)	–0.2228	9
Lin. Apprx. ($n=7$)	(–1.3845, 2.6886)	–0.2207	13
Quad. Apprx. ($n=7$)	(–1.3119, 2.7512)	–0.1953	13
Lin. Apprx. ($n=9$)	(–1.3119, 2.7512)	–0.1953	17
Quad. Apprx. ($n=9$)	(–1.3119, 2.7512)	–0.1953	17

4.2 *Hypothetical Limit State Function with Eight Variables*

This example considers a hypothetical limit state function with mixed uncertain variables of the following form:

$$g(\boldsymbol{x}) = x_1 - \sqrt{\left(\frac{x_2 x_3 \sqrt{x_4}\,(x_5+x_6)}{x_7} + x_8\right)^2 + x_3 x_5^2} \; . \quad (24)$$

The properties of mixed uncertain variables are presented in Table 2. The target reliability index is set as $\beta_t = 4.75$ (which corresponds to a failure probability $P_F = 1.0171\times10^{-6}$). The limit state function given in Equation 24 is approximated using first-order HDMR by deploying n sample points along each of the variable axis and taking respectively the mean values and nominal values of the random and fuzzy variables as reference point $\boldsymbol{c}$.

Table 2. Properties of the uncertain variables

Variable	Mean	SD	Type	Fuzzy
x_1	1.12	0.2	Normal	
x_2	1.77	0.2	Normal	
x_3	-	-	-	[0.32, 0.41, 0.50]
x_4	1	0.2	Normal	
x_5	-	-	-	[0.16, 0.20, 0.24]
x_6	-	-	-	[0.07, 0.10, 0.13]
x_7	-	-	-	[1.0, 1.5, 2.0]
x_8	0.1	0.01	Normal	

The approximated limit state function is divided into two parts, one with only the random variables along with the value of the constant part, and the other with the fuzzy variables. The joint membership function of the fuzzy part of approximated limit state function is obtained using suitable transformation of the fuzzy variables.

Unlike the case when all uncertain variables are random, the presence of fuzzy variables along with random variables, leads to the membership function of MPP (x_1^*, x_2^*, x_4^* and x_8^*) instead of having a unique value at the target reliability index. This is similar to the concept applied in reliability analysis, where the presence of interval variables results in the bounds on reliability or membership function of reliability, instead of having a unique value.

Figures 2(a)–2(d) respectively show the membership functions of x_1^*, x_2^*, x_4^* and x_8^* values at the target reliability index estimated by the proposed method using linear and quadratic approximations. The effect of number of sample points is studied by varying n from 3 to 9 in obtaining the membership function of design variables. In Figures 2(a)–2(d), it can be observed that the membership functions of x_1^*, x_2^*, x_4^* and x_8^* values estimated by the proposed method using $n = 7$ and 9 are overlapping each other.

4.3 *12-Stories and 3-Bays Linear Frame Structure*

A linear frame structure with twelve stories and three bays as shown in Figure 3 is considered. The cross sectional areas A_1 to A_5 are assumed to be log-normally distributed random variables with mean values of 0.25, 0.16, 0.36, 0.2 and 0.15, and standard deviation values of 0.025, 0.016, 0.036, 0.02 and 0.015 respectively. The horizontal load P is treated as fuzzy with a triplet of $[22.5, 30, 37.5]$.

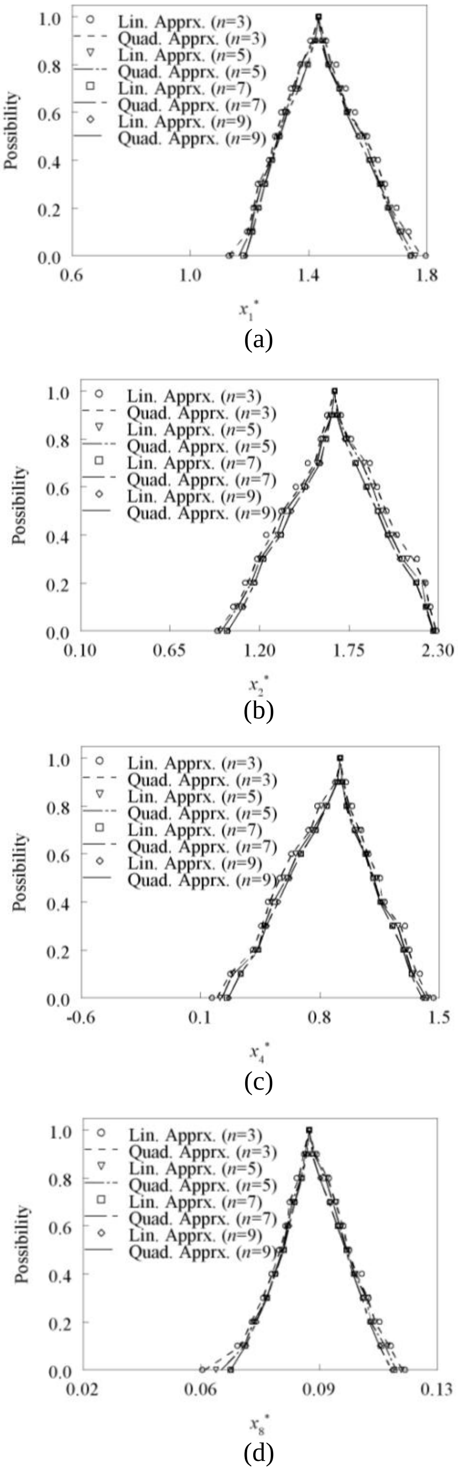

Figure 2. Membership function of design variables: (a) x_1^*; (b) x_2^*; (c) x_4^*; and (d) x_8^*

The sectional moments of inertia are expressed as $I_i = \alpha_i A_i^2$ ($i = 1-5$, $\alpha_1, \alpha_2, \alpha_3 = 0.083$, $\alpha_4 = 0.267$,

$\alpha_5 = 0.2$). The Young's modulus is treated as deterministic. Element types are indicated in Figure 3. In this study, the functional relationship to define the horizontal displacement at the top of the frame is:

$$g(A_i, P) = \Delta_{\lim} - u_h; \;\; i = 1-5, \qquad (25)$$

where $\Delta_{\lim}$ is taken as 0.1 m. Our interest is to find $A_i^*(i=1-5)$, such that the target reliability index $\beta_t = 1.4391$ ($P_F = 7.5058 \times 10^{-2}$) is achieved. The implicit limit state function given in Equation 25 is approximated using first-order HDMR by deploying n sample points along each of the variable axis and taking respectively the mean values and nominal values of the random and fuzzy variables as reference point $\boldsymbol{c}$.

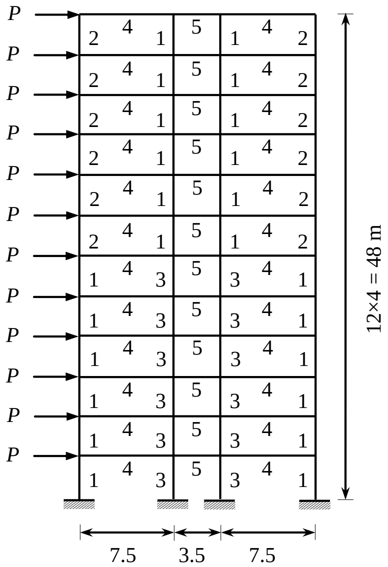

Figure 3. 12-story frame structure

The approximated limit state function is divided into two parts, one with only the random variables along with the value of the constant part, and the other with the fuzzy variables. The joint membership function of the fuzzy part of approximated limit state function is obtained using suitable transformation of the fuzzy variables.

Using the proposed inverse reliability method in conjunction with linear and quadratic approximations membership functions of $A_i^*(i=1-5)$ values at the target reliability index are estimated, and shown in Figures 4(a)–4(e). In addition, the effect of number of sample points is studied by varying n from 3 to 9, and it can be observed from the Figures 4(a)–4(e) that the membership functions of $A_i^*(i=1-5)$ values estimated by the proposed method using $n = 7$ and 9 are overlapping each other.

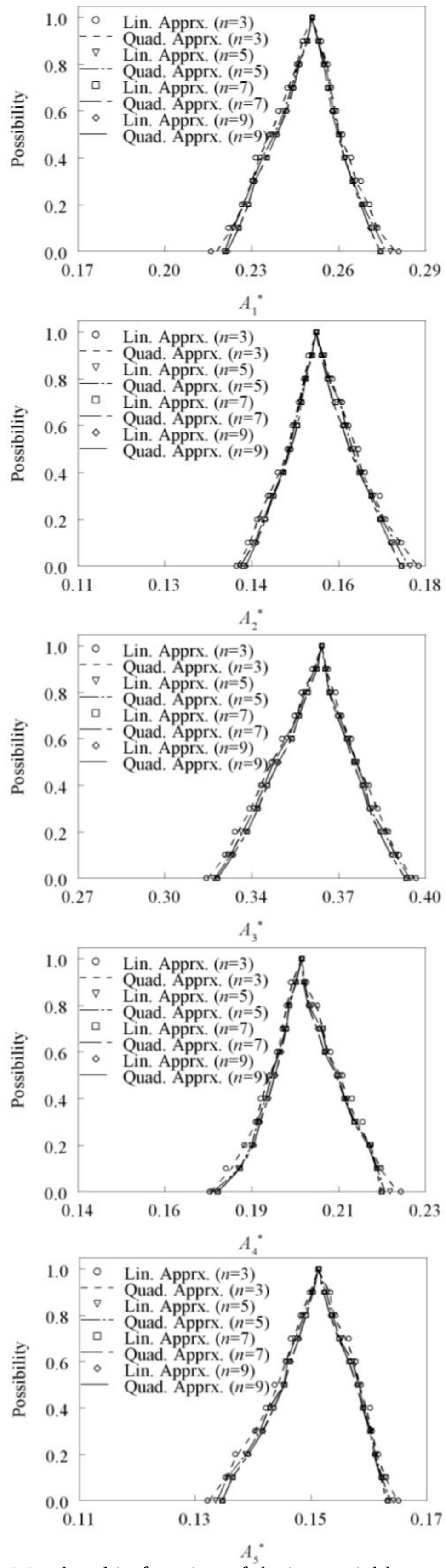

Figure 4. Membership function of design variables: (a) A_1^*; (b) A_2^*; (c) A_3^*; (d) A_4^*; and (e) A_5^*

6 SUMMARY AND CONCLUSIONS

An efficient, accurate, robust solution procedure alternative to existing inverse reliability methods is proposed for nonlinear problems with implicit response functions, which can be used to determine multiple unknown design parameters such that prescribed reliability indices are attained in the presence of mixed uncertain variables. The proposed method avoids the requirement of the derivatives of the response functions with respect to the uncertain variables. The proposed computational procedure involves three steps: (i) probability of failure calculation using High Dimensional Model Representation (HDMR) for the limit state function approximation, transformation technique to obtain the contribution of the fuzzy variables to the convolution integral, and fast Fourier transform for solving the convolution integral, (ii) reliability index update, and (iii) most probable point update. The methodology developed is versatile, hence can be applied to highly nonlinear or multi-parameter problems applicable involving any number of fuzzy variables and random variables with any kind of distribution. The accuracy and efficiency of the proposed method is demonstrated through three numerical examples. In addition, a parametric study is conducted with respect to the number of sample points used in approximation of HDMR component functions and its effect on the estimated solution is investigated. Very small number of sample points should be avoided as approximation may not capture the nonlinearity outside the domain of sample points and thereby affecting the estimated solution.

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