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Closed form solutions for element matrices of 4-node rectangular plate element using IFM

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ABSTRACT: This paper presents closed form solutions for equilibrium and flexibility matrices of the Mindlin-Reissner theory based 4-node rectangular plate bending element using Integrated Force Method (IFM). Use of closed form solutions of equilibrium and flexibility matrices reduce the computational time significantly and more suitable for the plate bending problems with square/rectangular boundaries. Large number of standard square/rectangular plate bending benchmark problems have been analyzed for central deflections and moments using the presented closed form solutions. Results are compared with those of similar displacement based plate bending elements available in the literature. The results are also compared with the exact solutions.

1 INTRODUCTION

Closed form solutions for element equilibrium and flexibility matrices of 4-node(MRP4) rectangular plate bending element are presented in this paper. The Mindlin-Reissner plate theory has been employed in the formulation as it accounts for the effect of shear deformation and the same model can be used for the analysis of both thin and moderately thick plate problems. The closed form solutions are more suitable for the analysis of thin/moderately thick plate problems with square / rectangular boundaries. Although the applications of square / rectangular plate bending elements are limited in practice, generally closed form solutions of equilibrium and flexibility matrices produce, in general, more accurate results in considerably less time compare to those obtained using numerical methods.

Extensive research efforts are spent in modeling the behavior of the elements and later deriving the matrices which represent their characteristic behavior in the finite element method of analysis. The various matrices are formed with interpolation functions for displacement and sometimes force distribution within or on the boundary of the element. Later on algebraic manipulations, including differentiation and integration, are performed on describing characteristics of the element stiffness, flexibility and equilibrium matrices. As the number of degrees of freedom of the element increases, the algebraic manipulations become huge and intractable. Therefore automatic generation and closed form of these matrices have been attempted by several researchers like Luff, et al. [1], Gunderson, et al. [2], Cecchi et al. [3], Noor, et al [4], Hoa et al [5], Chang et al. [6], Yew et al.[7], Eriksson et al.[8], Nagabhushanam et al.[9]. Closed form of stiffness matrices for a four node quadrilateral element and commonly used hybrid finite elements are developed by Griffiths [10] and Lee et al, [11]. Rectangular finite element formulation with its applications are given by Oztorun[12].

Analogous to development of closed form solutions or automatic generation of stiffness matrices in the displacement-based finite element method as cited above, the IFM is also in need of development of closed form solutions of element equilibrium and flexibility matrices, and compatibility conditions for analyzing civil, mechanical and aerospace engineering structures. In this direction, Nagabhushanam, et al, [13], developed a general purpose program to generate compatibility matrix for the Integrated Force Method. Automatic generation of sparse and banded compatibility matrix using the Integrated

Force Method is presented by Nagabhushanam, et al,[14]. In this paper, IFM has been used to obtain closed form solutions for equilibrium and flexibility matrices of the Mindlin-Reissner theory based 4-node rectangular plate bending element for the analysis of thin (t/L ≤ 0.01, where t= thickness of plate and L = span of plate) and moderately thick (0.01 < t/L ≤ 0.2) square/rectangular plate problems.

The Integrated Force Method (IFM) is a new novel matrix formulation developed by Patnaik[15] for the analysis of civil, mechanical and aerospace engineering structures. In general equilibrium equations, compatibility conditions are to be satisfied in addition to the constitutive relations which describe the material behavior while analyzing the structural mechanics problems. In this method all internal forces of the structure are treated as unknown variables and computed by simultaneously imposing equilibrium equations and compatibility conditions. The IFM integrates the system equilibrium equations and the global compatibility conditions in a fashion paralleling approaches in continuum mechanics (example, the Beltrami - Michel formulation of elasticity 16]. IFM is based on variational principles [17] and its stationary condition of the functional yields the equilibrium equations, compatibility and natural boundary conditions.

Unlike classical force method of analysis, the IFM is independent of redundants and the basic determinate structure. The IFM provides a natural way of integrating the equilibrium equations and the compatibility conditions while performing structural analysis. IFM requires explicit generation of compatibility conditions for skeletal as well as continuum structures. The advantages of IFM compare to displacement-based finite element method are reported in the reference [18]. In this paper, closed form solutions for equilibrium and flexibility matrices of 4-node rectangular plate bending element(MRP4) for analyzing the thin/moderately thick plate bending problems using IFM is presented. The Mindlin-Reissner theory has been employed in the plate bending formulation which accounts for the shear deformation. Three degrees of freedom namely a transverse displacement w and two rotations θ_x, θ_y are considered at each node of 4-node element. The shear correction factor as suggested by Reissner[19] has been considered in the formulation. Displacement and stress-resultants fields are chosen over the element and the corresponding element equilibrium and flexibility matrices are obtained in closed form using exact integration. To validate these closed form equilibrium and flexibility matrices of the element MRP4, standard square/rectangular plate bending benchmark problems are analyzed for central deflections and moments. The results obtained by the closed form solutions are compared with those obtained using displacement -based four node quadrilateral elements available in the literature [20]. Results are also compared with the exact solutions. The closed form solutions presented in this paper produce excellent results for both thin and moderately thick plate bending problems with square / rectangular boundaries.

2 FORMULATION OF ELEMENT EQUILIBRIUM AND FLEXIBILITY MATRICES

Formulation of equilibrium and flexibility matrices for Mindlin - Reissner theory based plate bending elements is explained: In the Mindlin - Reissner theory where a line that is straight and normal to mid-surface of the un-deformed plate remain straight but not necessarily normal to the mid-surface of the deformed plate. This leads to the following definition of the displacement components u, v, w in the x,y,z Cartesian coordinates system.

$$u = -z\theta_x(x, y);$$

$$v = -z\theta_y(x, y);$$

$$w = w(x, y) \tag{1}$$

where

x, y are coordinates in the reference mid-surface

z is the coordinate through the thickness t with $-t/2 \le z \le t/2$

w is the transverse (lateral) displacement

θ_x, θ_y represent the rotations of the normal in x-z and y-z planes respectively.

Engineering strains for the Mindlin-Reissner theory can be written as

$$\{\varepsilon\} = -z\{k_1\} \tag{2}$$

where

$$\{\varepsilon\} = [\varepsilon_x \quad \varepsilon_y \quad \gamma_{xy} \quad \gamma_{yz} \quad \gamma_{zx}]^T;$$

$$\{k_1\} = \left[\frac{\partial\theta_x}{\partial x} \quad \frac{\partial\theta_y}{\partial y} \quad \frac{\partial\theta_x}{\partial y} + \frac{\partial\theta_y}{\partial x} \quad \frac{\theta_y - \frac{\partial w}{\partial y}}{z} \quad \frac{\theta_x - \frac{\partial w}{\partial x}}{z} \right]^T$$

The stress - strain relations for an isotropic two-dimensional plate material is given by

$$\{\sigma\} = [C_{con}]\{\varepsilon\} \tag{3}$$

where $\{\sigma\} = [\sigma_x \quad \sigma_y \quad \tau_{xy} \quad \tau_{yz} \quad \tau_{xz}]^T$

= Vector of stress components

$\{\varepsilon\} = [\varepsilon_x \quad \varepsilon_y \quad \gamma_{xy} \quad \gamma_{yz} \quad \gamma_{xz}]^T$

= Vector of strain components

[Ccon] = constitutive matrix

$$= \frac{E}{(1-\nu^2)}\begin{bmatrix} 1 & \nu & 0 & 0 & 0 \\ \nu & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{(1-\nu)}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{(1-\nu)}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{(1-\nu)}{2} \end{bmatrix}$$

E = Young's modulus; ν = Poisson's ratio

The stress-resultants $\{M\}$ can be expressed as

$$\{M\} = \int_{-t/2}^{t/2} [\sigma_x z \quad \sigma_y z \quad \tau_{xy} z \quad \tau_{yz} \quad \tau_{zx}]^T dz \tag{4}$$

where $\{M\} = [M_x \quad M_y \quad M_{xy} \quad Q_y \quad Q_x]^T$

= Vector of stress - resultants

$$\{\sigma_r\} = \left[\sigma_x \quad \sigma_y \quad \tau_{xy} \quad \frac{\tau_{yz}}{z} \quad \frac{\tau_{xz}}{z}\right]^T$$

Equations 2, 3 and 4 yield the moment-curvature relations as

$$\{M\} = [C_1]\{k\} \tag{5}$$

where $\{k\}$ = Vector of Curvatures

$$= \left[\frac{\partial \theta_x}{\partial x} \quad \frac{\partial \theta_y}{\partial y} \quad \frac{\partial \theta_x}{\partial y} + \frac{\partial \theta_y}{\partial x} \quad \theta_y - \frac{\partial w}{\partial y} \quad \theta_y - \frac{\partial w}{\partial y}\right]^T$$

$[C_1]$ = matrix relating stress resultants to curvatures

From the equation (5), the curvature moment relation becomes

$$\{k\} = [C_1]^{-1}\{M\} = [H]\{M\} \tag{6}$$

where $[H] = [C_1]^{-1}$ is the matrix relating curvatures to stress-resultants and it can be written with Reissner's shear correction factor of 5/6 as:

$$[H] = \frac{1}{D_1}\begin{bmatrix} 1 & -\nu & 0 & 0 & 0 \\ -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 2(1+\nu) & 0 & 0 \\ 0 & 0 & 0 & \frac{t^2(1+\nu)}{5} & 0 \\ 0 & 0 & 0 & 0 & \frac{t^2(1+\nu)}{5} \end{bmatrix} \tag{7}$$

where $D_1 = Et^3/12$;

t = thickness of the plate;

The Strain energy U_p of the plate in bending is given by

$$U_p = \iint 1/2\{k\}^T\{M\}dxdy \tag{8}$$

For a discrete plate bending element the {M} and {k} can be expressed in terms of assumed stress-resultant and displacement fields respectively in the matrix form as

$$\{M\} = [\psi]\{F_e\} \tag{9}$$

$$\{k\} = [D_{op}][\phi_1]\{\alpha\} = [D_{op}][\phi]\{X_e\} \tag{10}$$

where

$[\psi]$ = matrix of polynomial terms for stress-resultant fields

$\{F_e\}$ = vector of force components of the discrete element

$[\phi_1]$ = matrix of polynomial terms for displacement fields

$[\phi] = [\phi_1][A]^{-1}$

$[A]$ = matrix formed by substituting the coordinates of the element nodes into the polynomial of displacement fields

$\{\alpha\}$ = coefficients of the displacement field polynomial

$\{X_e\} =$ vector of displacements of the discrete element

$\lfloor D_{op} \rfloor$ = differential operator matrix

$$= \begin{bmatrix} 0 & \frac{\partial}{\partial x} & 0 \\ 0 & 0 & \frac{\partial}{\partial y} \\ 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \\ -\frac{\partial}{\partial y} & 0 & 1 \\ -\frac{\partial}{\partial x} & 1 & 0 \end{bmatrix}$$

Substituting equations 9 and 10 into the equation 8, the strain energy for the discrete element can be expressed as

$$U_p = \frac{1}{2}\{X_e\}^T [B_e]\{F_e\} \tag{11}$$

where $[B_e]$ represents the element equilibrium matrix and is given by

$$[B_e] = \iint \lfloor \phi \rfloor^T \lfloor D_{op} \rfloor^T \lfloor \psi \rfloor dxdy \tag{12}$$

Using the equation 7, the complementary strain energy of the element is written as

$$\{^B \quad \}^T [\quad]\{\quad \}$$

$$\{\ _e\}^T [\ _e]\{\ _e\}$$

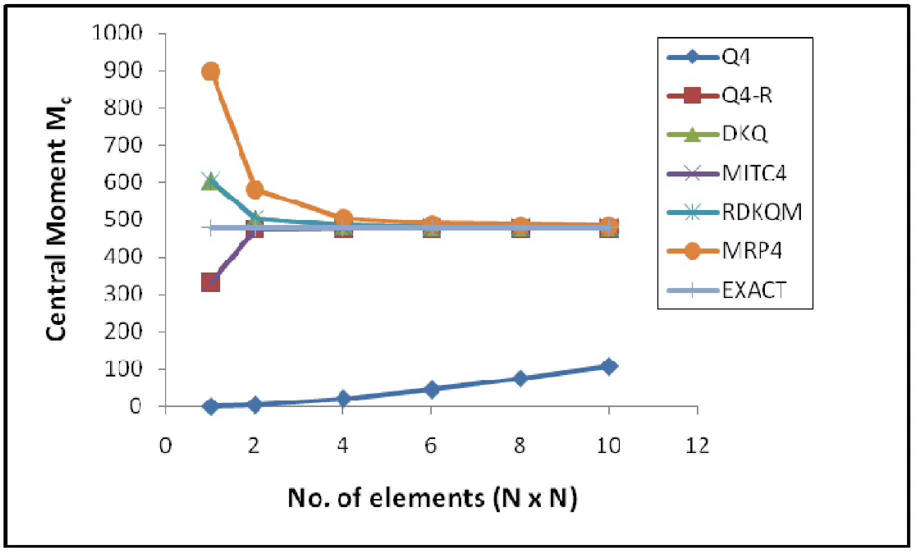

Figure 3: Central moment for a simply supported square thin (t/L=0.01) plate with uniform load

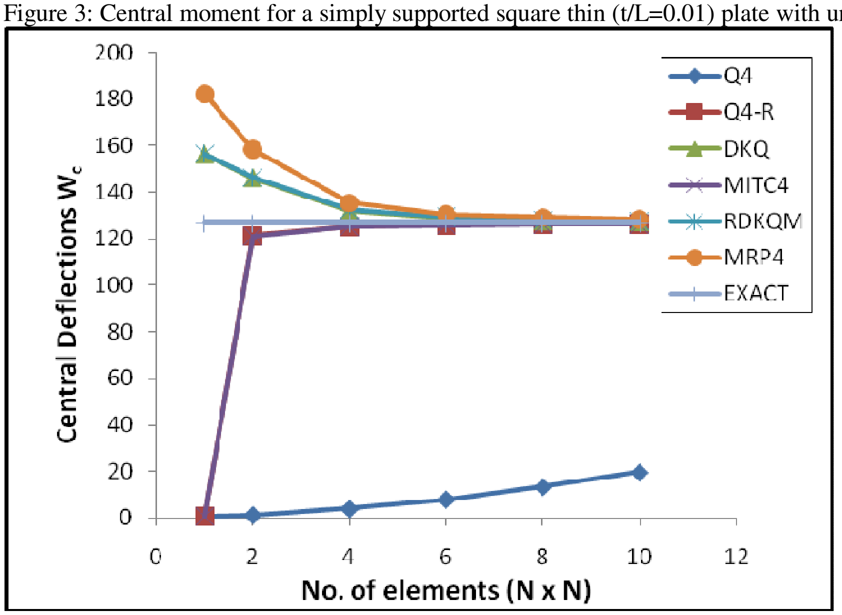

Figure 4: Central deflection for a clamped square thin (t/L=0.01) plate with uniform load

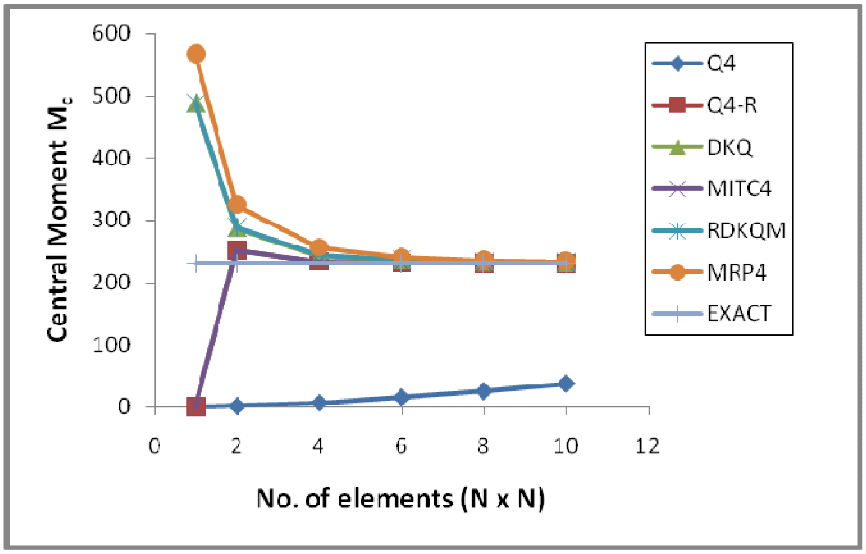

Figure 5: Central moment for a clamped square thin (t/L=0.01) plate with uniform load

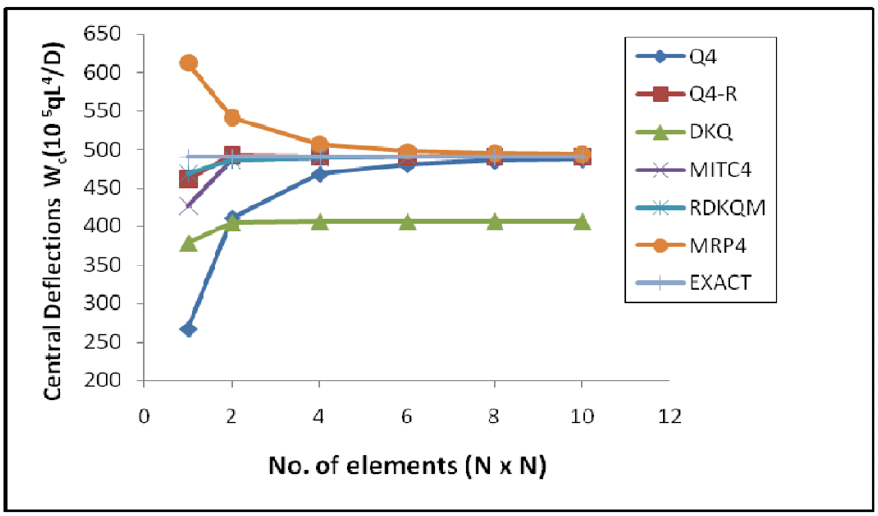

Figure 6: Central deflection for a simply supported square thick (t/L=0.2) plate with uniform load

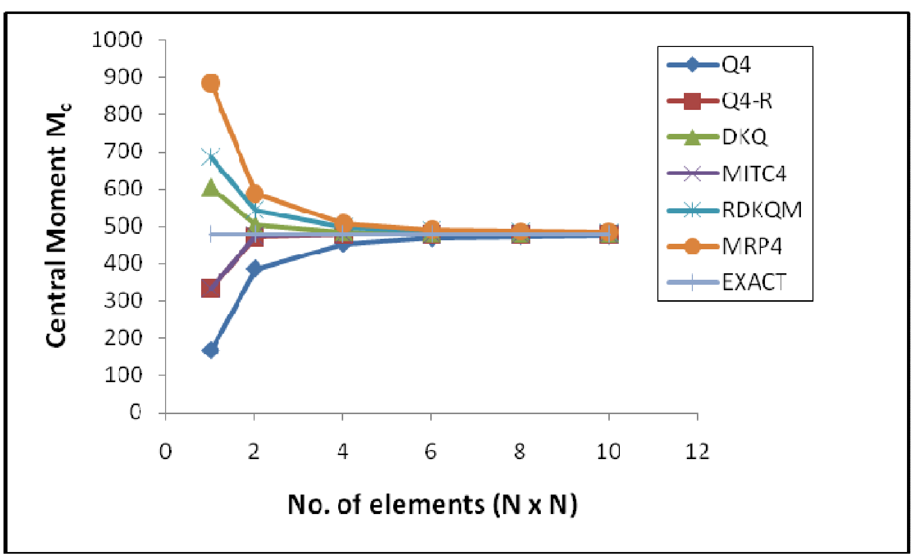

Figure 7: Central moment for a simply supported square thick (t/L=0.2) plate with uniform load

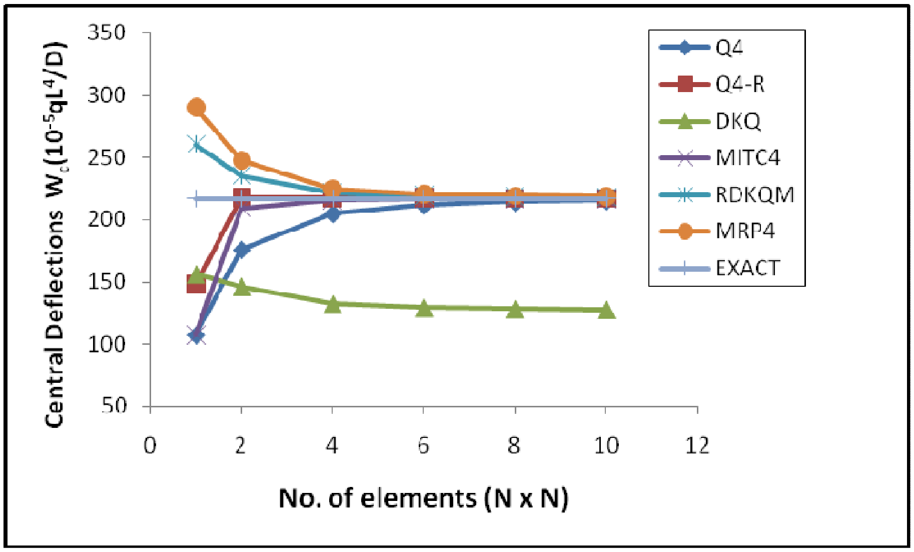

Figure 8: Central deflection for a clamped square thick plate (t/L=0.2) with uniform load

4 CONCLUSIONS

Closed form solutions for equilibrium and flexibility matrices of Mindlin-Reissner theory based 4-node rectangular plate bending element MRP4 are presented. These matrices are validated by analyzing standard plate bending benchmark problems to obtain central deflections and moments. The results are compared with those obtained from displacement-based 4-node similar elements. The results are also compared with the exact solutions. The results obtained using these closed form solutions are continuously converging towards exact solutions for various mesh sizes in both thin and thick plate bending situations. Therefore these closed form solutions can be used to analyze both thin and moderately thick plate bending problems with square / rectangular boundaries.

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