eJSE International

Electronic Journal of Structural Engineering, 9 (2009)

Nonlinear Analysis of Reinforced Concrete Frames by a Combined Method

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ABSTRACT: A more realistic and hence nonlinear analysis of reinforced concrete structures is becoming increasingly important. A combination of the displacement method, the transfer matrix method, and a cross-section module is suggested which leads to an effective analysis method for reinforced and prestressed concrete frames. The combined method considers both material and geometrical nonlinearities including large displacements and rotations. The computation of the system is incrementally and iteratively carried out by the displacement method. At element level, an extended transfer matrix method is used. Thus neither displacement nor force shape functions are required. Instead, the axial strain and curvature distributions along the element are segmentally approximated by polynomials. The transfer matrix method provides both the element forces and the element stiffness matrix. It is recursively applied to the deformed element, which is discretised into individual segments whose number and lengths depend on the stiffness gradient. The cross section module is based on cross-sectional integration. It takes into account nonlinear material behaviour including cracking, softening and yielding of reinforcement. The combined method is presented for plane frames but can be extended to spatial systems.

1 INTRODUCTION

New and improved construction methods render possible bolder and more slender concrete structures. A basic prerequisite for this development is the availability of realistic and efficient computational tools. Material and geometrical nonlinearities have to be taken into account on grounds of safety and economy. The possibility of nonlinear analysis based on predefined realistic stress-strain curves has already been incorporated into several building codes (see e.g. CEP-FIP (1993), DIN (2001), MCPRC (2002)). Nevertheless, the nonlinear material behaviour is usually considered only in the cross-sectional design of individual members in concrete structures. A coherent analysis and design concept is only arrived at, however, when the internal forces are also determined in a nonlinear analysis. For instance, a nonlinear computation which accurately predicts deformations is essential for a safe design of slender concrete columns. Regarding statically indeterminate systems, a consideration of the internal force redistribution due to nonlinearities can prevent local underestimation of internal forces, deformations and ductility requirements. Furthermore, internal force redistribution can lead to a better utilisation of the load-bearing capacity of the system, possibly resulting in a more economical design. Regarding the effect of restraining action (temperature, shrinkage, etc.), the computed internal forces will generally be smaller when cracking, softening, and creep are considered.

The current research on numeric concrete modeling focuses on micro-models including the formulation of three-dimensional material models and three-dimensional finite elements. With respect to the investigation of frames, macro-models with one-dimensional elements are preferred because they are easier to establish and to interpret and the cost of computation is lower.

Spacone & El-Tawil (2004) give an overview of several frame elements for nonlinear analysis. The elements can be classified into displacement-based and force-based methods. Displacement-based elements use shape functions to describe the displacement along the element depending on the nodal displacements. They are derived from a weak equilibrium condition and satisfy displacement continuity at the element ends but not necessarily equilibrium. Force-based elements, also referred to as flexibility-based, assume interpolation function for the distribution of the internal forces depending on the nodal forces. They have turned out to be very robust and need fewer degrees of freedom for comparable accuracy. The main challenge is the integration of such elements in a nonlinear analysis pro-

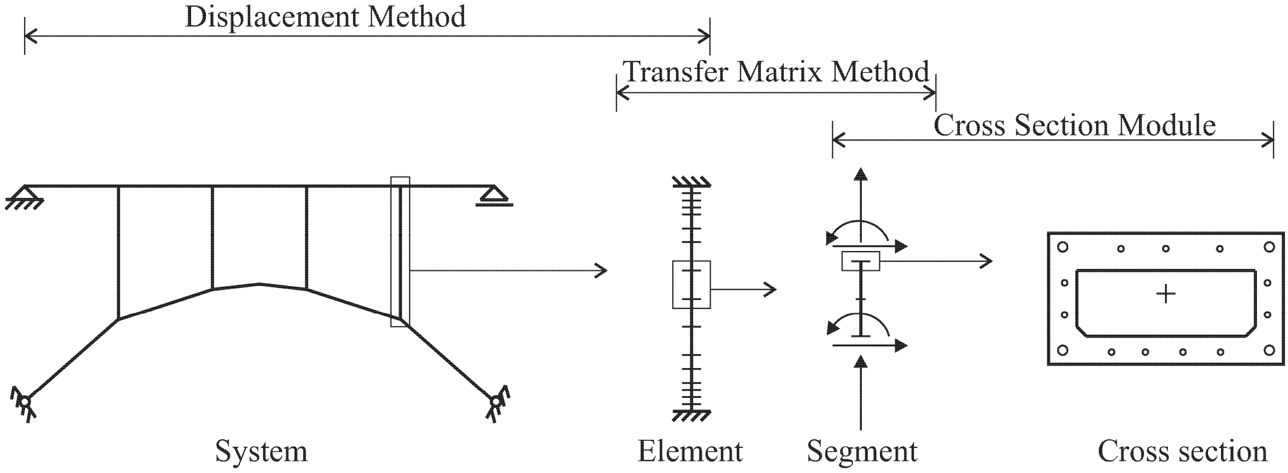

Figure 1. Levels of the combined method.

gram. A complex procedure is required for deriving the element stiffness matrix and element resisting forces. Spacone et al. (1996) propose an iterative element state determination by adjusting the element forces until the predetermined displacements are achieved. The stiffness matrix follows from the inverted flexibility matrix. Neuenhofer & Filippou (1997) present a method for directly determining the element state thus avoiding an iterative procedure at element level. By means of interpolation functions for the curvature, that approach is adapted to geometrical nonlinearity considering small deformations but is restricted to material linearity (Neuenhofer & Filippou 1998).

The approach presented here allows for geometrical nonlinearity including large displacements and rotations and material nonlinearity. Neither displacement nor force shape functions are required. Instead, the axial strain and curvature distributions along the element are segmentally approximated by polynomials. The respective advantages of three methods are combined. Olsen (1986) and Pfeiffer (2004) explore similar ideas and can be considered precursors to the approach presented here. It is suitable for the computation of general reinforced or prestressed concrete frames, but also for steel-concrete composite frames and others. In contrast to lumped models, which concentrate the nonlinearity at element ends, the combined method represents a distributed approach.

At system level, the computation is incrementally and iteratively carried out using the displacement method. The advantage of the displacement method lies in the ease of representation of a frame's topology by elements. System nodes are mainly defined at the beamand column end and connection points, and at cross-sectional changes. At element level, an extended transfer matrix method is used (Wallmichrath & Starossek 2004). In the transfer matrix scheme, the state variables are recursively transferred over a chosen number of discrete segments from one end of the element to the other. The state variables comprise the internal forces and the displacement quantities. The partitioning of an element into segments depends on the local stiffness gradient. Based on given element end node displacements, which follow from the first level computation at system level, the internal forces at the element end nodes and at the segments are determined. Having obtained the internal forces, the tangent stiffness matrix is calculated using difference quotients. The remaining unbalance forces at the element end nodes enter into the next iteration step at system level and decrease with each further iteration step until achieving convergence.

Material nonlinearity along the element is considered within the cross section module via uniaxial fiber stress-strain relations. Curvature and axial strain for a given set of internal forces are iteratively determined by cross-sectional integration. In this way, the strain state corresponds exactly to the internal forces. The axial strain and curvature are the basis for the computation of displacements over the segment length by the transfer matrix method. This procedure is contrary to the classic displacement-based method, where the strain state is obtained via the derivations of the displacement shape functions. The reader is referred to Figure 1 for a schematic description of the hierarchical structure of the method. For a detailed discussion see Wallmichrath (2007).

2 ASSUMPTIONS

- All cross section dimensions are small compared to the beam length.
- The system is plane.
- Cross sections are non-deformable and symmetric to the system plane.
- The cross sections remain plane, shear deformations are neglected (Bernoulli hypothesis).

- A perfect bond of all cross sectional components is assumed.
- Strain states and normal stress states are assumed to be uniaxial to the beam's longitudinal axis.
- Material behaviour is time-invariant.
- Strains are sufficiently small.
- Large large displacements and rotations are allowed.
- Loads are static and act in the plane of the structure.

An extension to space frames and time-dependent effects is possible.

3 CROSS SECTION MODULE

In compliance with material nonlinearity, the cross section module supplies the strain state for any given set of internal forces applied to arbitrary polygonal shaped cross sections. Both non-prestressed as well as prestressed reinforcement are considered. Regarding the method presented here, the transfer matrix method (superscript "TMM" in the equation below) delivers the internal forces. The strain state is iteratively determined by solving the inverse problem. The resulting internal forces of a given strain state can be determined directly via integration of the corresponding stresses over the cross section. Stress integration is carried out according to Rotter (1985) and Fafitis (2001), respectively, and applies to biaxial bending with normal force. With regard to the plane problems presented here, however, integration is limited to uniaxial bending (*M*) with normal force (*N*). The double integral over a cross section is, according to Green's theorem, transformed into a line integral around the edges of the cross section. Line integration is carried out numerically by Gauss-Legendre quadrature. The portion of reinforcement regarding the internal forces is determined through the resulting single forces.

On the basis of suitable initial values for the strain state the latter is improved in every single step j using the Newton-Raphson method

$$\begin{bmatrix}\varepsilon_0\\ \kappa\end{bmatrix}^{(j+1)} = \begin{bmatrix}\varepsilon_0\\ \kappa\end{bmatrix}^{(j)} = \left(\begin{bmatrix}\frac{\partial N}{\partial \varepsilon_0} & \frac{\partial N}{\partial \kappa}\\ \frac{\partial M}{\partial \varepsilon_0} & \frac{\partial M}{\partial \kappa}\end{bmatrix}^{(j)}\right)^{-1} \cdot \left(\begin{bmatrix}N\\ M\end{bmatrix}^{\mathrm{TMM}} - \begin{bmatrix}N\\ M\end{bmatrix}^{(j)}\right) \quad (1)$$

Suitable initial values for the curvature $\kappa^{(0)}$ and the strain at centroid $\varepsilon_0^{(0)}$ can be taken from the nearby cross section or the last step of a preceding iteration within the scope of the transfer matrix method. The partial derivatives are replaced by difference quotients. Cross section iteration will end if either a previously defined convergence criterion is fulfilled or if it is impossible to find a match after a defined number of iterations. The latter will occur if the outer internal forces exceed the load-bearing capacity of the cross section. The determination of the cross-sectional strain state constitutes approximately 90 % of total computation time.

4 TRANSFER MATRIX METHOD

The transfer matrix method is appropriate to the analysis of unbranched frame structures. However, due to the combination with the displacement method arbitrary branched frame structures can be analysed. An extensive presentation of the transfer matrix method is provided by Pestel & Leckie (1963). The application of the method for the determination of the stiffness matrix and nodal load vector of non-prismatic elastic frame elements is presented by Luo et al. (2007). The transfer matrix method has also practical implementations in various fields including bridge construction (Rosignoli 1999), high-rise buildings (Akintilo & Syngelakis 1989) as well as in vibration control (Yang & Samali 1983).

The fundamental idea behind the method is the transfer of the state variables segment by segment from an element's beginning to its end in compliance with the loads, stiffnesses and intermediate conditions. For simple systems, a transfer can be formulated via a transfer matrix. Material und geometrical nonlinearities are captured in a recursive procedure. Within the framework of the combined method, the transfer matrix method is applied at element level.

4.1 *Application to frame elements*

For the transfer matrix method, the element is subdivided into n_p segments of length l_p. The adaptive discretisation depends on the local stiffness gradient and produces no additional global degrees of freedom. Element loads can quite simply be included in the procedure. A separate post-computation is not required.

Starting with the left element end, internal forces and displacements are successively determined for each single segment by using equilibrium conditions or by integration of the corresponding axial strains and curvatures, respectively. Since the equilibrium condition has to be determined by taking the deformations into account, the single segments are moved and rotated before their state variables are determined. Each segment is tangent to the end point of the preceding segment. The first segment is tangent to the left element end node. Figure 2 shows a part of the element with the translated and rotated segment p (full line). In the lower part of the figure the element is depicted as a dotted line in its unde-

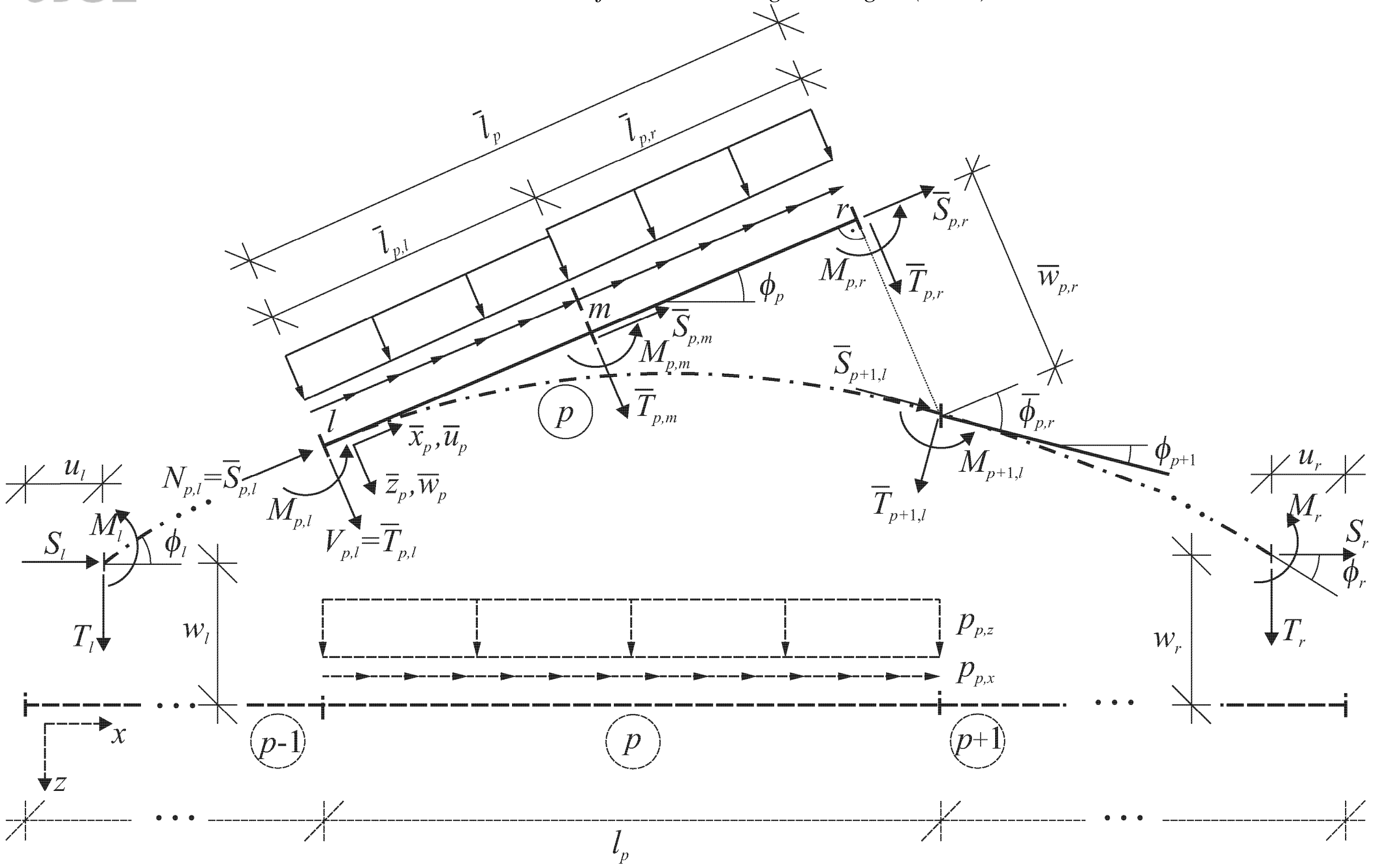

Figure 2. Recursive transfer matrix method.

formed position with the line loads. In its deformed position the element is shown as a dash-dotted line.

Due to the geometrical nonlinearity and segmentation of the element, one has to distinguish between different reference systems regarding internal forces and displacements. Internal forces which relate to the undeformed element axis are described as S and T for axial force and transverse force, respectively. The values u, w, and ϕ stand, respectively, for longitudinal and transverse displacements and rotation. Corresponding values with a horizontal line are defined in the local reference system of the translated and rotated but undeformed segment. Internal forces which are oriented relatively to the deformed line are referred to in the usual way as normal force N and shear force V. The respective forces can be converted through an orthogonal transformation. Regarding the rotation invariant bending moment M, differentiation and transformation are unnecessary.

Because of the geometrical and material nonlinearities, a direct computation of a segment's state variables is no longer possible. But when the transverse and longitudinal displacements are applied sequentially, one after the other, the procedure becomes a straightforward one. First the axial elongation of the segment is determined. Subsequently the transverse displacements are computed taking into account the changed length of the segment. The curvature distribution over the segment's length is quadratically approximated with the help of interpolation nodes at both segment's ends and center. A two-stage computation of the transverse displacement is used to identify the interaction of the deformation and the internal forces. In the first stage, transverse displacements are determined in the axially stretched segment according to the first-order theory (i.e. neglecting the P-Δ effect). The second stage takes into account the deformations determined in the preceding stage while computing internal forces and deformations anew. From these deformations a amplification factor ensues which is used to determine the final transverse displacements, rotations and bending moments.

4.2 *Segment level*

It is assumed that the internal forces at the left segment end $\bar{\mathbf{f}}_{p,l}=[\bar{S}_{p,l},\bar{T}_{p,l},M_{p,l}]^T$ are known, since they follow from the preceding segment or the element end node. The cross section module provides the curvature $\kappa_{p,l}$ and the axial strain $\varepsilon_{0,p,l}$ consistent with the internal forces. The internal forces are always converted into the local normal force N and the corresponding bending moment M. By means of the axial strain the new length of the left segment part is determined: $\bar{l}_{p,l}=(1+\varepsilon_{0,p,l})\ l_p/2$. The axial strain includes the strain of the normal force and, due to material nonlinearity, the bending moment as well. Considering the new length of the left segment half and the transformed loads, the internal forces at the segment center $\bar{\mathbf{f}}_{p,m}=[\bar{S}_{p,m},\bar{T}_{p,m},M_{p,m}]^T$ are determined using equilibrium conditions. Starting with these internal forces the right segment part is similarly computed yielding $\bar{\mathbf{f}}_{p,r}=[\bar{S}_{p,r},\bar{T}_{p,r},M_{p,r}]^T$. For three points along the segment the internal forces and, after application of the cross section module, the corresponding curvatures of the first computation stage are known. The distribution of the curvature over the segment length $\kappa^{(I)}(x;^{-})$ is approximated using a quadratic parabola. By integration of this function the rotations and transverse displacements are determined for both the segment's center and its end according to

$$\bar{\phi}^{(I)}_{p,m} = -\int_0^{\bar{l}_{p,l}} \kappa^{(I)}(\bar{x})\, d\bar{x}_p, \qquad \bar{\phi}^{(I)}_{p,r} = -\int_0^{\bar{l}_p} \kappa^{(I)}(\bar{x})\, d\bar{x}_p \tag{2}$$

$$\bar{w}^{(I)}_{p,m} = -\iint_0^{\bar{l}_{p,l}} \kappa^{(I)}(\bar{x})\, d\bar{x}_p d\bar{x}_p, \quad \bar{w}^{(I)}_{p,r} = -\iint_0^{\bar{l}_p} \kappa^{(I)}(\bar{x})\, d\bar{x}_p d\bar{x}_p \tag{3}$$

where $\bar{l}_p$ is the new stretched segment length. In the second stage, the same procedure is used to determine internal forces, the corresponding curvatures, the resulting transverse displacement of the segment end $\bar{w}^{(II)}_{p,r}$, and take into account the deformations found during the first computation stage. The new displacements would lead to further changes in the internal forces. When this procedure is continued one obtains an infinite sequence of displacements. Assuming affinity and a constant ratio between two succeeding displacement increments, the final displacement corresponds to a geometric series, which converges to

$$\bar{w}_{p,r} = \frac{1}{1-\beta}\bar{w}^{(I)}_{p,r}, \quad \beta = \frac{\bar{w}^{(II)}_{p,r} - \bar{w}^{(I)}_{p,r}}{\bar{w}^{(I)}_{p,r}} \text{ for } |\beta| < 1. \tag{4}$$

The same factor is applied to the rotations and bending moments

$$\bar{\phi}_{p,r} = \frac{1}{1-\beta}\bar{\phi}^{(I)}_{p,r}, \quad M_{p,r} = \frac{1}{1-\beta}\bar{M}^{(I)}_{p,r}. \tag{5}$$

The changes in axial deformation through geometrical nonlinearity are neglected. The longitudinal displacements are therefore determined directly from change in length of the segment according to the first computation stage $\bar{u}_{p,r} \approx \bar{u}^{(I)}_{p,r} = \bar{l}_p - l_p$. Together with the axial force $S;^{-}_{p,r}$ and transverse force $T;^{-}_{p,r}$, all internal forces at the segment end are known.

4.3 *Load and displacement transfer between segments*

According to the equilibrium of forces at the connection node of the segments, the transfer to the subsequent segments is carried out using an orthogonal rotation matrix

$$\bar{\mathbf{f}}_{p+1,l} = -\mathbf{R}_{\bar{\phi}_{p,r}} \cdot \bar{\mathbf{f}}_{p,r} \tag{6}$$

$$\begin{bmatrix} \bar{S}_{p+1,l} \\ \bar{T}_{p+1,l} \\ M_{p+1,l} \end{bmatrix} = -\begin{bmatrix} \cos\bar{\phi}_{p,r} & \sin\bar{\phi}_{p,r} & 0 \\ -\sin\bar{\phi}_{p,r} & \cos\bar{\phi}_{p,r} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \bar{S}_{p,r} \\ \bar{T}_{p,r} \\ M_{p,r} \end{bmatrix}.$$

Adding the fictitious segments 0 and n_p+1 includes the internal forces at the left and right element end node to a recursive formulation

$$\begin{aligned} p = 0:\ & \mathbf{f}_{0,r} = -\mathbf{f}_l = -[S_l, T_l, M_l]^T,\ \phi_{0,r} = \phi_l; \\ p = n_p:\ & \mathbf{f}_{n_p+1,l} = -\mathbf{f}_r = -[S_r, T_r, M_r]^T,\ \phi_{n_p,r} = -\phi_r. \end{aligned} \tag{7}$$

Together, Equations 6 and 7 provide for a complete, recursive expression for the internal forces of the right element end node. The computation of the displacements is carried out by summation of the deformations of the segments within the element

$$\mathbf{v}_r = \mathbf{v}_l + \sum_{p=1}^{n_p} \mathbf{v}_{p,r}, \quad \begin{bmatrix} u_r \\ w_r \\ \phi_r \end{bmatrix} = \begin{bmatrix} u_l \\ w_l \\ \phi_l \end{bmatrix} + \sum_{p=1}^{n_p} \begin{bmatrix} u_{p,r} \\ w_{p,r} \\ \phi_{p,r} \end{bmatrix}. \tag{8}$$

The portion of the respective segments to the displacement values comprises of the rigid body motion $\mathbf{v}_{p,\mathrm{RMB}}$ and the displacement components of the segment deformation $\bar{\mathbf{v}}_{p,r}$. These have to be transformed to the reference system of the undeformed element axis

$$\mathbf{v}_{p,r} = \mathbf{R}_{\phi_p} \cdot \bar{\mathbf{v}}_{p,r} + \mathbf{v}_{p,\mathrm{RBM}} \tag{9}$$

$$\begin{bmatrix} u_{p,r} \\ w_{p,r} \\ \phi_{p,r} \end{bmatrix} = \begin{bmatrix} \cos\phi_p & \sin\phi_p & 0 \\ -\sin\phi_p & \cos\phi_p & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \bar{u}_{p,r} \\ \bar{w}_{p,r} \\ \bar{\phi}_{p,r} \end{bmatrix} - l_p \begin{bmatrix} 1-\cos\phi_p \\ \sin\phi_p \\ 0 \end{bmatrix}.$$

The computation of deformations regarding the respective segments is carried out within the scope of the theory of small deformations. However, the transformation in Equation 9 considers large rotations.

5 ELEMENT LEVEL

5.1 *Resisting forces*

At the element level, the internal forces at the element end nodes are determined using the transfer matrix method. The first level computation of the system using the general displacement method (superscript "DM") provides for the displacement of system's nodes. These displacements are transformed to the element reference system, and the rigid body motion is deducted. The task is now to find the internal forces at the left element node $\mathbf{f}_l$ which lead to the given relative element node displacements $\mathbf{v}^{\mathrm{DM}}_{\mathrm{rel}}$. The computation is incrementally carried out using the Newton-Raphson method. On the basis of suitable initial values for $\mathbf{f}_l$ and with the help of the transfer matrix method, the resulting displacements at the right element end node are determined in each step. If the left element end node is chosen as the reference node for displacement values, the displacement values of the right node are equivalent to the relative displacements $\mathbf{v}_r^{(i)} = \mathbf{v}_{\text{rel}}^{(i)}$. With regard to iteration, the following applies:

$$\mathbf{f}_l^{(i+1)} = \mathbf{f}_l^{(i)} + \mathbf{k}_{T,lr}^{(i)} \cdot \left(\mathbf{v}_{\text{rel}}^{\text{DM}} - \mathbf{v}_{\text{rel}}^{(i)}\right), \qquad (10)$$

$$\begin{bmatrix} S_l \\ T_l \\ M_l \end{bmatrix}^{(i+1)} = \begin{bmatrix} S_l \\ T_l \\ M_l \end{bmatrix}^{(i)} + \left(\begin{bmatrix} \dfrac{\partial u_r}{\partial S_l} & \dfrac{\partial u_r}{\partial T_l} & \dfrac{\partial u_r}{\partial M_l} \\ \dfrac{\partial w_r}{\partial S_l} & \dfrac{\partial w_r}{\partial T_l} & \dfrac{\partial w_r}{\partial M_l} \\ \dfrac{\partial \phi_r}{\partial S_l} & \dfrac{\partial \phi_r}{\partial T_l} & \dfrac{\partial \phi_r}{\partial M_l} \end{bmatrix}^{(i)} \right)^{-1} \cdot \left(\begin{bmatrix} u_{\text{rel}} \\ w_{\text{rel}} \\ \phi_{\text{rel}} \end{bmatrix}^{\text{DM}} - \begin{bmatrix} u_{\text{rel}} \\ w_{\text{rel}} \\ \phi_{\text{rel}} \end{bmatrix}^{(i)} \right).$$

The matrix $\mathbf{k}_{T,lr}^{(i)}$ describes the linearised relation between the left internal forces and the right displacements. The matrix mathematically corresponds to the inverse Jacobian matrix $\partial \mathbf{v}_r / \partial \mathbf{f}_l$. Mechanically, this matrix forms a tangent stiffness matrix or, before inversion, a tangent flexibility matrix. The matrix components are approximately computed using a difference quotient. This is accomplished by varying each of the left node internal forces by a small amount and then determining the resulting deformation values at the right node using the transfer matrix method.

5.2 *Tangent stiffness matrix*

The element tangent stiffness matrix $\mathbf{k}_T$ describes the locally linearised relation between the six nodal forces $\delta\mathbf{f}$ and the six element degrees of freedom $\delta\mathbf{v}$. The components are arranged according to the left and right node

$$\mathbf{k}_T \cdot \delta\mathbf{v} = \partial\mathbf{f}, \quad \left[\begin{array}{c|c} \mathbf{k}_{T,ll} & \mathbf{k}_{T,lr} \\ \hline \mathbf{k}_{T,rl} & \mathbf{k}_{T,rr} \end{array}\right] \cdot \begin{bmatrix} \delta\mathbf{v}_l \\ \delta\mathbf{v}_r \end{bmatrix} = \begin{bmatrix} \delta\mathbf{f}_l \\ \delta\mathbf{f}_r \end{bmatrix}, \qquad (11)$$

$$\left[\begin{array}{ccc|ccc} k_{T,ll}^{11} & k_{T,ll}^{12} & k_{T,ll}^{13} & k_{T,lr}^{11} & k_{T,lr}^{12} & k_{T,lr}^{13} \\ k_{T,ll}^{21} & k_{T,ll}^{22} & k_{T,ll}^{23} & k_{T,lr}^{21} & k_{T,lr}^{22} & k_{T,lr}^{23} \\ k_{T,ll}^{31} & k_{T,ll}^{32} & k_{T,ll}^{33} & k_{T,lr}^{31} & k_{T,lr}^{32} & k_{T,lr}^{33} \\ \hline k_{T,rl}^{11} & k_{T,rl}^{12} & k_{T,rl}^{13} & k_{T,rr}^{11} & k_{T,rr}^{12} & k_{T,rr}^{13} \\ k_{T,rl}^{21} & k_{T,rl}^{22} & k_{T,rl}^{23} & k_{T,rr}^{21} & k_{T,rr}^{22} & k_{T,rr}^{23} \\ k_{T,rl}^{31} & k_{T,rl}^{32} & k_{T,rl}^{33} & k_{T,rr}^{31} & k_{T,rr}^{32} & k_{T,rr}^{33} \end{array}\right] \cdot \left[\begin{array}{c} \delta u_l \\ \delta w_l \\ \delta\phi_l \\ \hline \delta u_r \\ \delta w_r \\ \delta\phi_r \end{array}\right] = \left[\begin{array}{c} \delta S_l \\ \delta T_l \\ \delta M_l \\ \hline \delta S_r \\ \delta T_r \\ \delta M_r \end{array}\right].$$

Submatrix $\mathbf{k}_{T,lr}$ links the three right displacements with the three left internal forces. It can be taken from the last step of the preceding internal force iteration. Submatrix $\mathbf{k}_{T,rl}$ is determined in the same way as $\mathbf{k}_{T,lr}$. Since all element forces are known from the preceding internal force iteration, the determination of matrix $\mathbf{k}_{T,rl}$ is limited to the application of internal force variation on the right edge. Accordingly, the transfer matrix method has to be formulated from right to left. To provide a symmetric tangent stiffness matrix a positive and a negative variation is applied. The two resulting matrices are averaged.

The columns of an element stiffness matrix can be interpreted as reaction forces of a fixed end beam which result from imposed unit displacement at the corresponding degree of freedom. Due to submatrices $\mathbf{k}_{T,rl}$ and $\mathbf{k}_{T,lr}$ three of the reaction forces per matrix column are known. Therefore, the remaining three are determined using equilibrium conditions. Mechanically, the resulting deformed shape represents the negative influence line of the corresponding reaction force. As an example, Figure 3 illustrates the negative influence line for the right bending moment.

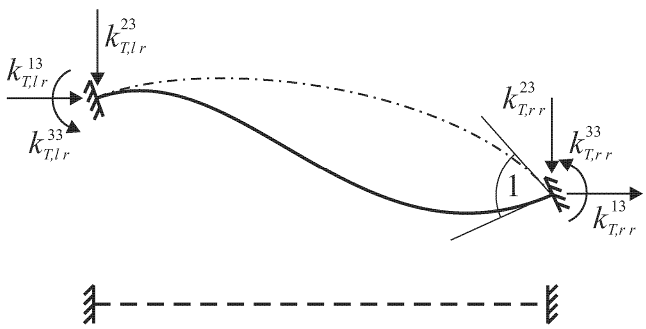

Figure 3. Negative influence line for the bending moment at the right end of the deformed element.

6 SYSTEM LEVEL

Equilibrium at the system nodes is formed according to the system's degrees of freedom. Computation is incrementally and iteratively carried out using the displacement method. The system tangent stiffness matrix, and the system load vector are assembled from the corresponding element matrices and vectors, respectively. The load is increased step by step until total load is reached. Regarding the single load steps, the iterative system computation is carried out using the Newton-Raphson method. The initial values for the first load step are taken from a linear elastic computation according to first-order theory. For a detailed description of this method, the reader may e.g. refer to Belytschko et al. (2000). A post-computation is not required since the transfer matrix method has already provided for the state values of the last step for all segments.

7 EXAMPLE

For an application of the method, the 4.5 m long column used in test II.1 by Fouré (1978) is investigated. An eccentric axial load was applied until failure in a short-time test. Dimensions and reinforcement of the column, which had pinned support at both ends can be seen in Figure 4. By use of symmetry, the one half of the system is modeled with one

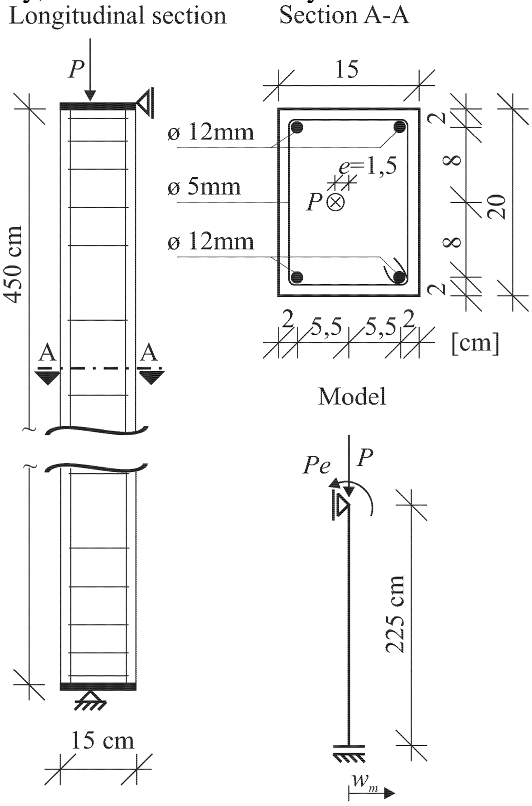

Figure 4. Experimental Setup.

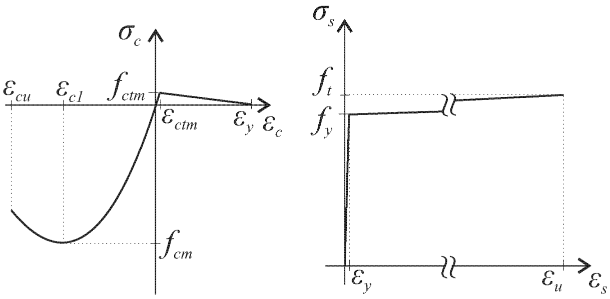

Figure 5. Assumed stress-strain relation for concrete and reinforcement.

element. The eccentric load is realised by simultaneously applying the axial load P and the moment Pe. Material properties which are not directly given by Fouré (1978) are determined according to DIN (2001).

The assumed stress-strain curve of concrete in compression corresponds to the rational function given in DIN (2001), and it is assumed to be bilinear in tension (Fig. 5). The stress-strain relationship of the reinforcement steel is assumed bilinear with strain hardening. For a list of material properties, the reader is referred to Table 1.

Table 1. Material properties of concrete and reinforcement.

Concrete	Reinforcement
f_{cm} = 38.3 N/mm²	f_y = 465 N / mm²
E_{cm} = 32,000 N/mm²	E_s = 20,3000 N/mm²
f_{ctm} = 2.9 N/mm²	f_t = 511.5 N/mm²
ε_{c1} = -2.3 mm/m	ε_u = 25 mm/m

In Figure 6 the load P is plotted versus the transverse displacement w_m at mid-height of the column. It can be seen that the result of the combined-method analysis closely follows the plot of the test data and reproduces the ultimate load with good accuracy.

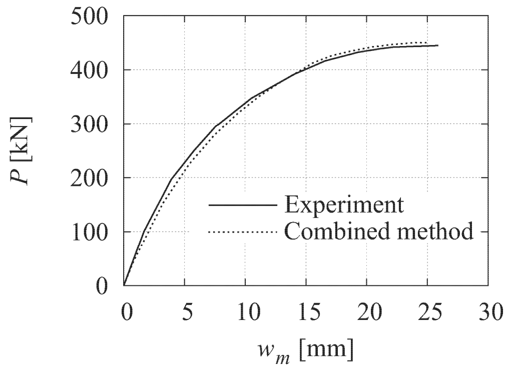

Figure 6. Load-displacement plot.

8 CONCLUSIONS

An approach for the nonlinear analysis of plane reinforced concrete frames is presented. Both material and geometrical nonlinearities including large large displacements and rotations are considered. The approach is a combination of the displacement method and the transfer matrix method. Material nonlinearities are efficiently taken care of by a cross section module, whereby the strain state corresponds exactly to the internal forces. Due to the use of the transfer matrix method this combined approach dispenses with displacement and force shape functions. The distributions of curvature and axial strain are segmentally approximated by polynomials. The numerical analysis of a slender column and the comparison with test data show good accuracy.

This approach can be extended to time-dependent effects. Creep and shrinkage models have recently been incorporated into the existing cross-section module (Wallmichrath 2007). An extension to space frames demands a sophisticated upgrading of the cross-section module to six internal forces (Löhning et al. 2007), the computation of the spatial displacement values, and a consideration of interaction between the spatial internal forces (Schenk et al. 2007).

APPENDIX

$\mathbf{f}$ = vector of internal forces
$\mathbf{k}_T$ = element tangent stiffness matrix

$\mathbf{k}_{T,lr}$	=	submatrix of stiffness matrix
$k_{T,lr}^{23}$	=	matrix coefficient
l	=	length of segment
M	=	bending moment
n_p	=	number of segments
N	=	normal force
$\mathbf{R}_\phi$	=	rotation matrix corresponding to ϕ
S	=	axial force
T	=	transverse force
u	=	longitudinal displacement
$\mathbf{v}$	=	displacement vector
V	=	shear force
w	=	transverse displacement
x,z	=	coordinate
β	=	ratio between deformation increments
ε_0	=	strain at centroid
κ	=	curvature
ϕ	=	rotation
subscript		
l,m,r	=	left, middle, right
p	=	segment number
rel	=	difference between element nodes
superscript		
(i), (j)	=	iteration step
(I), (II)	=	computation stage
TMM, DM	=	value of transfer matrix, displacement method

Symbols with a bar are defined in the section reference system.

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