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Structural Failure Probability Estimation Using HDMR and FFT

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ABSTRACT: This paper presents a new and alternative method based on High Dimensional Model Representation (HDMR) and fast Fourier transform (FFT) to estimate the structural failure probability of structural systems subject to random loads, material properties and geometry. The proposed methodology is based on the limit state/performance function approximation and the convolution theorem to estimate the structural failure probability. The limit-state function is obtained by linear approximation of the first-order HDMR component functions at the most probable failure point, and the convolution integral is solved efficiently using the FFT technique. The proposed technique estimates the failure probability accurately with significantly less computational effort compared to the direct Monte Carlo simulation. The accuracy and efficiency of the proposed method is demonstrated through numerical examples involving implicit performance functions.

Keywords: *Structural Reliability, High Dimensional Model Representation, Fast Fourier Transform and Failure Probability.*

1 INTRODUCTION

The structural reliability problem consists in determining the probability that a structure exceeds a threshold limit, defined by a limit state/performance function influenced of several random parameters. Symbolically, the reliability problem can be stated as,

$$P_F \equiv P\left(g\left(\boldsymbol{x}\right)\le 0\right)=\int_{g(\boldsymbol{x})\le 0} p_X\left(\boldsymbol{x}\right)\ d\boldsymbol{x}\,, \qquad (1)$$

where $\boldsymbol{x}=\{x_1,x_2,\cdots,x_N\}$ represents the N–dimensional random variables of the model under consideration; $g\left(\boldsymbol{x}\right)$ is the limit state/performance function, such that $g\left(\boldsymbol{x}\right)\le 0$ represents the failure domain; and $p_X\left(\boldsymbol{x}\right)$ is the joint probability density function of the input random variables.

Two class of approaches are commonly available for estimating the failure probability P_F, which can be labeled as gradient-based and simulation-based methods. In the first case there is a need of estimating the gradient of the limit state/performance function in a relevant point around which the largest concentration of the probability mass in the failure region can be found. These methods are popularly called as first- or second-order reliability method (FORM/SORM). A crucial point in their application is the need of knowing the limit state function explicitly. But, in reality, the limit state/performance functions are implicit nature and highly nonlinear. Therefore, a detailed finite element (FE) modeling of the structure is necessary in combination with reliability analysis tools. FE methods for linear and nonlinear structures in conjunction with FORM/SORM have been successfully applied for structural reliability computations. But, such methods are effective in evaluating very small probabilities of failure for small-scale problems. In regard to the large-scale problems, merging of FORM/SORM, with commercial FE programs is not straightforward especially when the nonlinear problems are addressed.

In contrast to the gradient-based methods, simulation-based methods hinge upon the creation of generating synthetic set of basic random variables samples and simulating the actual limit state/performance function repeated times. The major disadvantages of the Monte Carlo simulation (MCS) are that the results are of a statistical value and the random sampling error will produce inaccuracy within the results. The importance sampling technique, a commonly used variance reduction technique, requires an appropriate importance sampling function in order to take full advantage of this method. As an alternative, simulation techniques have been combined with the response surface method in order to reduce the number of simulations, in such a way that the response surface acts as a surrogate to the finite element solver for obtaining most of the samples.

Wu and Torng (1990) used a quadratic approximation of the limit state/performance function at the

most probable point (MPP) and adopted fast Fourier transform (FFT) to estimate the failure probability. This method can produce accurate results for some problems but since the failure probability calculation is entirely dependent on the approximation at the MPP, higher order function approximation methods would be a better choice compared to a quadratic approximation. The essence of FFT based probability analysis was demonstrated by Sakamoto et al. (1997). In order to use FFT, the limit state/performance function must be separable. Therefore, Sakamoto et al., (1997) adopted the response surface concept to get separable and closed form expression of the implicit limit state/performance function. Penmetsa and Grandhi (2003), and Adduri and Penmetsa (2007) implemented two-point adaptive nonlinear approximation (Wang, and Grandhi, 1995) to construct the approximate limit state/performance function and used FFT technique to estimate the failure probability.

Present effort is based on (1) HDMR approximation of the original implicit limit state/performance function and (2) the convolution theorem to estimate the structural failure probability. The proposed method estimates the joint probability density of the approximate limit state/performance function using FFT technique and the failure probability can be estimated by integration over failure domain. Comparisons have been made with conventional approximate methods (FORM/SORM) and direct MCS method to evaluate the accuracy and the computational efficiency of the present method.

2 CONCEPT OF HDMR AND ITS USEFULNESS FOR FAILURE PROBABILITY ESTIMATION

In recent years there have been efforts to develop efficient methods to approximate multivariate functions in such a way that the component functions of the approximation are ordered starting from a constant and gradually approaching to multivariance as we proceed along the terms like first-order, second-order and so on. One such method is high dimensional model representation (HDMR) (Rabitz et al., 1999). HDMR is a general set of quantitative model assessment and analysis tools for capturing the high-dimensional relationships between sets of input and output model variables. It is a very efficient formulation of the system response, if higher order variable correlations are weak, allowing the physical model to be captured by the first few lower order terms. Practically for most well-defined physical systems, only relatively low order correlations of the input variables are expected to have a significant effect on the overall response. HDMR expansion utilizes this property to present an accurate hierarchical representation of the physical system.

Degree of accuracy of structural reliability estimation depends on the accurate representation of the limit state/performance function. Computational complexity for the generation of response surface arises due to increase in number of input variables, while using conventional response surface in conjunction with design of experiments. The concept of HDMR expansions is introduced here for the purpose of representing the response function most accurately and efficiently, when the number of input variables is large.

Let the N–dimensional vector $\boldsymbol{x}=\{x_1,x_2,\ldots,x_N\}$, represent the input variables of the model under consideration, and the response variable as $g(\boldsymbol{x})$. Since the influence of the input variables on the response variable can be independent and/or cooperative, HDMR expresses the response $g(\boldsymbol{x})$ as a hierarchical correlated function expansion in terms of the input variables as,

$$g(\boldsymbol{x}) = g_0 + \sum_{i=1}^{N} g_i(x_i) + \sum_{1\le i_1<i_2\le N} g_{i_1 i_2}(x_{i_1},x_{i_2}) + \ldots + \sum_{1\le i_1<\ldots<i_l\le N} g_{i_1 i_2 \ldots i_l}(x_{i_1},x_{i_2},\ldots,x_{i_l}) + \ldots + g_{12\ldots N}(x_1,x_2,\ldots,x_N) \quad (2)$$

where g_0 is a constant term representing the zeroth-order component function or the mean response of $g(\boldsymbol{x})$. The function $g_i(x_i)$ is a first-order term expressing the effect of variable x_i acting alone, although generally nonlinearly, upon the output $g(\boldsymbol{x})$. The function $g_{i_1 i_2}(x_{i_1},x_{i_2})$ is a second-order term which describes the cooperative effects of the variables x_{i_1} and x_{i_2} upon the output $g(\boldsymbol{x})$. The higher order terms gives the cooperative effects of increasing numbers of input variables acting together to influence the output $g(\boldsymbol{x})$. The last term $g_{12\cdots N}(x_1,x_2,\cdots,x_N)$ contains any residual dependence of all the input variables locked together in a cooperative way to influence the output $g(\boldsymbol{x})$. Once all the relevant component functions in Equation 2 are determined and suitably represented, then the component functions constitute HDMR, thereby replacing the original computationally expensive method of calculating $g(\boldsymbol{x})$ by the computationally efficient model. Usually the higher order terms in Equation 2 are negligible such that HDMR with only low order correlations to second-order (Li et al., 2001), amongst the input variables are typically adequate in describing the output behavior. Therefore it is expected that HDMR expansion converges very rapidly.

Depending on the method adopted to determine the component functions in Equation 2 there are two particular HDMR expansions: ANOVA-HDMR and

cut-HDMR. ANOVA-HDMR is useful for measuring the contributions of the variance of individual component functions to the overall variance of the output. On the other hand, cut-HDMR expansion is an exact representation of the output $g(\boldsymbol{x})$ in the hyperplane passing through a reference point in the variable space.

With cut-HDMR method, first a reference point $\boldsymbol{c}=\{c_1,c_2,\cdots,c_N\}$ is defined in the variable space. In the convergence limit, cut-HDMR is invariant to the choice of reference point $\boldsymbol{c}$. In practice, $\boldsymbol{c}$ is chosen within the neighborhood of interest in the input space. The expansion functions are determined by evaluating the input-output responses of the system relative to the defined reference point $\boldsymbol{c}$ along associated lines, surfaces, subvolumes, etc. (i.e. cuts) in the input variable space. This process reduces to the following relationship for the component functions in Equation 2

$$g_0 = g(\boldsymbol{c}), \tag{3}$$

$$g_i(x_i) = g(x_i,\boldsymbol{c}^i) - g_0, \tag{4}$$

$$g_{i_1i_2}(x_{i_1},x_{i_2}) = g(x_{i_1},x_{i_2},\boldsymbol{c}^{i_1i_2}) - g_{i_1}(x_{i_1}) - g_{i_2}(x_{i_2}) - g_0, \tag{5}$$

where the notation $g(x_i,\boldsymbol{c}^i)=g(c_1,c_2,\ldots,c_{i-1},x_i,c_{i+1},\ldots,c_N)$ denotes that all the input variables are at their reference point values except x_i. The g_0 term is the output response of the system evaluated at the reference point $\boldsymbol{c}$. The higher order terms are evaluated as cuts in the input variable space through the reference point. Therefore, each first-order term $g_i(x_i)$ is evaluated along its variable axis through the reference point. Each second-order term $g_{i_1i_2}(x_{i_1},x_{i_2})$ is evaluated in a plane defined by the binary set of input variables x_{i_1},x_{i_2} through the reference point, etc. The process of subtracting off the lower order expansion functions removes their dependence to assure a unique contribution from the new expansion function.

Considering terms up to first-order in Equation 2 yields, respectively

$$g(\boldsymbol{x}) = g_0 + \sum_{i=1}^{N} g_i(x_i) + \mathrm{R}_2, \tag{6}$$

Substituting Equations 3 and 4 into Equation 6 leads to

$$g(\boldsymbol{x}) = \sum_{i=1}^{N} g(c_1,\ldots,c_{i-1},x_i,c_{i+1},\ldots,c_N) - (N-1)\,g(\boldsymbol{c}) + \mathrm{R}_2. \tag{7}$$

Now consider first-order approximation of $g(\boldsymbol{x})$, denoted respectively by

$$\tilde{g}(\boldsymbol{x}) \equiv g(x_1,x_2,\ldots,x_N) = \sum_{i=1}^{N} g(c_1,\ldots,c_{i-1},x_i,c_{i+1},\ldots,c_N) - (N-1)\,g(\boldsymbol{c}), \tag{8}$$

Comparison of Equations 7 and 8 indicates that the first-order approximation leads to the residual error $g(\boldsymbol{x})-\tilde{g}(\boldsymbol{x})=\mathrm{R}_2$, which includes contributions from terms of two and higher order component functions.

The notion of 0th, 1st, etc. in HDMR expansion should not be confused with the terminology used either in the Taylor series or in the conventional least-squares based response surface model. It can be shown that, the first order component function $g_i(x_i)$ is the sum of all the Taylor series terms which contain and only contain variable x_i. Hence first-order HDMR approximations should not be viewed as first-order Taylor series expansions nor do they limit the nonlinearity of $g(\boldsymbol{x})$. Furthermore, the approximations contain contributions from all input variables. Thus, the infinite number of terms in the Taylor series are partitioned into finite different groups and each group corresponds to one cut-HDMR component function. Therefore, any truncated cut-HDMR expansion provides a better approximation and convergent solution of $g(\boldsymbol{x})$ than any truncated Taylor series because the latter only contains a finite number of terms of Taylor series. Furthermore, the coefficients associated with higher dimensional terms are usually much smaller than that with one-dimensional terms. As such, the impact of higher dimensional terms on the function is less, and therefore, can be neglected. Compared with the FORM and SORM which retains only linear and quadratic terms, respectively, first-order HDMR provides more accurate response surface approximation $\tilde{g}(\boldsymbol{x})$ of the original implicit limit state/performance function $g(\boldsymbol{x})$.

3 PROBABILITY DENSITY AND CHARATERISTIC FUNCTIONS

The characteristic function, which is the Fourier transform of the marginal density, and the marginal density of a random variable Y are expressed as a pair of Fourier transforms (Lin, 1967) as follows:

$$p_Y(y) = \int_{-\infty}^{\infty} M_Y(\theta)\, e^{-2\pi i\theta y}\, d\theta, \tag{9}$$

where $p_Y(y)$ and $M_Y(\theta)$ are the marginal density and characteristics function of Y, respectively. i de-

notes the imaginary number defined as $i=\sqrt{-1}$. These equations define the forward and the inverse Fourier transforms with $M_Y(\theta)$ as the forward Fourier transform of $p_Y(y)$, and $p_Y(y)$ as the inverse Fourier transform of $M_Y(\theta)$. The properties of the characteristic function $M_Y(\theta)$ can be summarized as:

(1) $|M_Y(\theta)| \le 1$, $M_Y(-\theta)=\overline{M_Y(\theta)}$, where $|\bullet|$ and $\bar{\bullet}$ are the absolute value and the complex conjugate of $\bullet$, respectively.

(2) The characteristic function of a random variable $X=aY+b$ is expressed as

$$M_X(\theta)=e^{2\pi i\theta y}M_Y(a\theta). \tag{10}$$

(3) The characteristic function of a random variable *Y*, which is the sum of statistically independent random variables $Y_1,Y_2,\cdots,Y_N$ is given by the product of the characteristic function of each random variable $M_{Y_1}(\theta),M_{Y_2}(\theta),\cdots,M_{Y_N}(\theta)$ as

$$M_Y(\theta)=M_{Y_1}(\theta)\times M_{Y_2}(\theta)\times\cdots\times M_{Y_N}(\theta). \tag{11}$$

4 ESTIMATION OF FAILURE PROBABILITY USING HDMR AND FFT

Concept of FFT can be applied to the problem if the limit state/performance function is in the form of a linear combination of independent variables and when either the marginal density or the characteristic function of each basic random variable is known. Even if the function of the basic variables is nonlinear, an appropriate transformation of the basic random variables could yield a linear function of independent random variables. To achieve this linear function, the original limit state/performance function can be approximated by using a first-order Taylor series expansion, but this gives very poor accuracy. If the second-order terms are considered in the approximation, the computational cost required for gradients calculation is very high. In the present study HDMR concepts are used to express the implicit limit state/performance function $g(\boldsymbol{x})$ which depends on $\boldsymbol{x}=\{x_1,x_2,\ldots,x_N\}\in\Re^N$, as a linear combination of lower order component functions. The steps involved in the proposed method for failure probability estimation as follows:

1. If $\boldsymbol{u}=\{u_1,u_2,\ldots,u_N\}^T\in\Re^N$ is the standard Gaussian space, let $\boldsymbol{u}^*=\{u_1^*,u_2^*,\ldots,u_N^*\}^T$ be the MPP or design point, determined by a standard nonlinear constrained optimization. The MPP has a distance β_{HL}, which is commonly referred to as the Hasofer–Lind reliability index. Construct an orthogonal matrix $\boldsymbol{R}\in\Re^{N\times N}$ whose *N*–th column is $\boldsymbol{\alpha}^*=\boldsymbol{u}^*/\beta_{HL}$, i.e., $\boldsymbol{R}=\left[\boldsymbol{R}_1\middle|\boldsymbol{\alpha}^*\right]$ where $\boldsymbol{R}_1\in\Re^{N\times N-1}$ satisfies $\boldsymbol{\alpha}^{*T}\boldsymbol{R}_1=0\in\Re^{1\times N-1}$. The matrix $\boldsymbol{R}$ can be obtained, for example, by Gram–Schmidt orthogonalization. For an orthogonal transformation $\boldsymbol{u}=\boldsymbol{R}\boldsymbol{v}$. Let $\boldsymbol{v}=\{v_1,v_2,\ldots,v_N\}^T\in\Re^N$ be the represent the rotated Gaussian space with the associated MPP $\boldsymbol{v}^*=\{v_1^*,v_2^*,\ldots,v_N^*\}^T$. The transformed limit state/performance functions $g(\boldsymbol{v})$ therefore maps the original function into rotated Gaussian space $(\boldsymbol{v})$.

First-order HDMR approximation of $g(\boldsymbol{v})$ in rotated Gaussian space $\boldsymbol{v}$ with $\boldsymbol{v}^*=\{v_1^*,v_2^*,\ldots,v_N^*\}^T$ as reference point leads to:

$$\begin{aligned}\tilde{g}(\boldsymbol{v}) &\equiv g(v_1,v_2,\ldots,v_N)\\ &=\sum_{i=1}^{N} g\left(v_1^*,\ldots,v_{i-1}^*,v_i,v_{i+1}^*,\ldots,v_N^*\right).\\ &-(N-1)\,g(\boldsymbol{v}^*)\end{aligned} \tag{12}$$

In addition to the chosen reference point (the MPP in the present work), the accuracy of first-order HDMR approximation of $g(\boldsymbol{v})$ in Equation 12 may depend on the orientation of the first $N-1$ axes. In the present work, the orientation is defined by the matrix $\boldsymbol{R}$. In the above expressions, the terms $g\left(v_1^*,\ldots,v_{i-1}^*,v_i,v_{i+1}^*,\ldots,v_N^*\right)$ are the individual response function and are independent of each other. Equation 12 can be rewritten as,

$$\tilde{g}(\boldsymbol{v})=a+\sum_{i=1}^{N} g\left(v_i,\boldsymbol{v}^{*i}\right), \tag{13}$$

where $a=-(N-1)\,g(\boldsymbol{v}^*)$.

2. An intermediate variables are defined as,

$$z_i=g\left(v_i,\boldsymbol{v}^{*i}\right). \tag{14}$$

The purpose of these new variables is to transform the approximate function $\tilde{g}(\boldsymbol{v})$ into the following form

$$\tilde{g}(\boldsymbol{v})=a+z_1+z_2+\cdots+z_N. \tag{15}$$

3. Due to rotational transformation, in $\boldsymbol{v}$-space component functions z_i in Equation 15 are expected to be linear or weakly nonlinear function of random variables v_i. In this work linear approximation of z_i are considered. Consider a linear approximation: $z_i=b_i+c_i\,v_i$, where coefficients $b_i\in\Re$ and $c_i\in\Re$ (non-zero) are obtained by are obtained by least-squares approximations from exact or numerically simulated conditional responses $\left\{g\left(v_i^1,\boldsymbol{v}^i\right),g\left(v_i^2,\boldsymbol{v}^i\right),\cdots,g\left(v_i^n,\boldsymbol{v}^i\right)\right\}^T$ at *n* sample points along the variable axis v_i. The least-squares approximation is chosen over interpolation, because the former minimizes the error when $n>2$.

4. Since v_i follows standard Gaussian distribution, marginal density of the intermediate variables z_i can be easily obtained by simple transformation (using chain rule).

$$p_{Z_i}(z_i) = p_{V_i}(v_i)\left|\frac{1}{dz_i/dv_i}\right|. \quad (16)$$

5. Now since the approximate function $\tilde{g}(\boldsymbol{v})$ is a linear combination of intermediate variables z_i, the joint density of $\tilde{g}(\boldsymbol{v})$, which is the convolution of the individual marginal density of the intervening variables z_i, can be expressed as follows:

$$p_{\tilde{G}}(\tilde{g}) = p_{Z_1}(z_1) \times p_{Z_2}(z_2) \times \ldots \times p_{Z_N}(z_N), \quad (17)$$

where $p_{\tilde{G}}(\tilde{g})$ represents joint density of the response function $\tilde{g}(\boldsymbol{v})$.

6. Applying FFT on both sides of Equation 17, leads to,

$$\begin{aligned} FFT\left[p_{\tilde{G}}(\tilde{g})\right] = {} & FFT\left[p_{Z_1}(z_1)\right] \\ & \times FFT\left[p_{Z_2}(z_2)\right] \\ & \times \ldots \times FFT\left[p_{Z_N}(z_N)\right] \end{aligned} . \quad (18)$$

7. By applying inverse FFT on both sides of Equation 18, joint density of the limit state/performance function $\tilde{g}(\boldsymbol{v})$ is obtained.

8. The probability of failure is given by the following equation with *t* being a threshold value

$$P_F = \int_{-\infty}^{t} p_{\tilde{G}}(\tilde{g})\, d\tilde{g}. \quad (19)$$

5 COMPUTATIONAL EFFORT

If *n* is the number of sample points taken along each of the variable axis and *s* is the order of the component function considered, starting from zeroth-order to *l*-th order, then the total number of function evaluations for HDMR approximation of the original limit state/performance function is given by, $\sum_{s=0}^{l}\left(N!(n-1)^s\right)/\left((N-s)!s!\right)$ which grows polynomially with *n* and *s*. As a few low order component functions of HDMR are used, the sample savings due to HDMR are significant compared to traditional sampling. Hence reliability analysis using HDMR relies on an accurate reduced model being generated with a small number of full model simulations. The reliability index β corresponding to the failure probability P_F can be obtained by

$$\beta = -\Phi^{-1}(P_F), \quad (20)$$

where $\Phi(\square)$ is the cumulative distribution function of a standard Gaussian random variable.

6 NUMERICAL EXAMPLES

Two numerical examples involving structural systems are presented to illustrate the performance of the present method in conjunction with HDMR based response surface generation. To evaluate the accuracy and the efficiency of the present method comparisons of the estimated failure probability P_F have been made with FORM/SORM and direct MCS. The coefficient of variation δ of the estimated failure probability P_F by direct MCS for the sampling size N_S considered, is computed using

$$\delta = \sqrt{\frac{(1-P_F)}{N_S P_F}}. \quad (21)$$

When comparing computational efforts by various methods in evaluating the failure probability P_F, the number of original limit state/performance function evaluations is chosen as the primary comparison tool in this paper. This is because of the fact that, number of function evaluations indirectly indicates the CPU time usage.

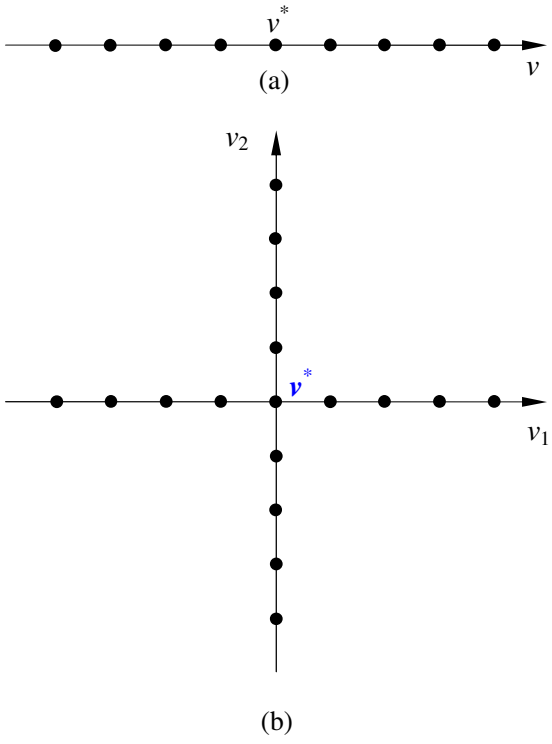


Figure 1. Sampling scheme for first-order HDMR; (a) For a function having one variable (v); and (b) For a function having two variables (v_1 and v_2)

For direct MCS, the number of original function evaluations is same as the sampling size. While evaluating the failure probability P_F through direct MCS, CPU time is more because it involves number of repeated actual finite-element analysis. To obtain

linear/quadratic approximation of the HDMR component functions, $n(=3,5,7 \text{ or } 9)$ uniformly distributed sample points $v_i^* - (n-1)/2$, $v_i^* - (n-3)/2$, …, v_i^*, …, $v_i^* + (n-3)/2$, $v_i^* + (n-1)/2$ are deployed along each of the variable axis v_i through the MPP. Thus total number of function evaluations required by the present method, in addition to those required for identification of the MPP is $(n-1)N$. Sampling scheme for response surface approximation of a function having one variable (v) and two variables (v_1 and v_2) is shown in Figures 1(a) and 1(b) respectively.

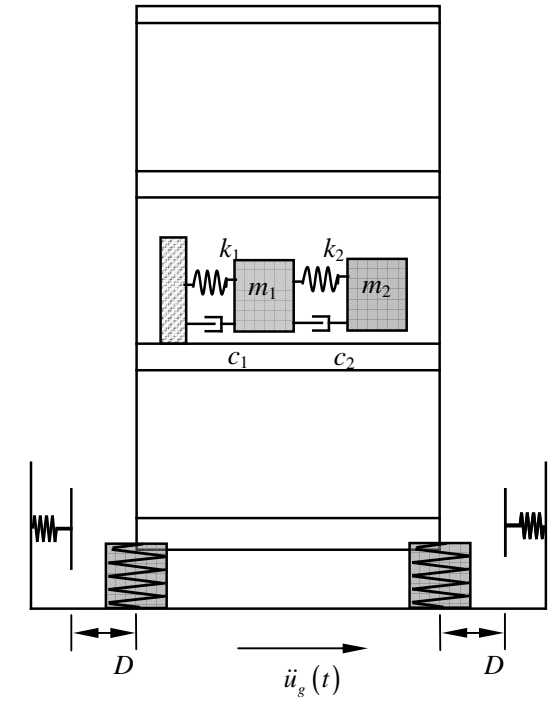
6.1 *Example 1: Dynamic Problem of Six Degrees of Freedom*

This example considers a four-storey building excited by a single period sinusoidal pulse of ground motion, studied by Gavin and Yau (2007). Figure 2(a) shows the four-storey building with isolation systems and Figure 2(b) presents the acceleration history. The building contains isolated equipment resting on the second floor. The motion of the ground floor is resisted mainly by base isolation bearings and if its displacement exceeds $D_c(=0.50\ m)$ then an additional stiffness force contributes to the resistance. Mass, stiffness and damping coefficient m_f, k_f and c_f, respectively at each floor are assumed to be same. There are two isolated masses, representing isolated, shock-sensitive equipment resting on the second floor. The larger mass $m_1(=500kg)$ is connected to the floor by a relatively flexible spring, $k_1(=2500N/m)$, and a damper, $c_1(=350N/m/s)$, representing the isolation system. The smaller mass $(m_2=100kg)$ is connected to the larger mass by a relatively stiff spring, $k_2(=10^5\ N/m)$, and a damper, $c_2(=200N/m/s)$, representing the equipment itself. All variables are assumed to be lognormal and independent.

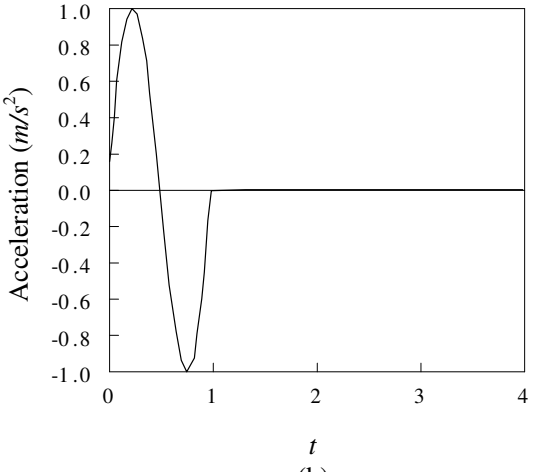
The statistical properties of the random variables are listed in Table 1. The limit state/performance function is defined by the combination of three failure modes leading to system failure and is the following form

$$g(\boldsymbol{x}) = 12.50\left(0.04 - \max_t \left|x_{f_i}(t) - x_{f_{i-1}}(t)\right|\right)_{i=2,3,4} + \left(0.50 - \max_t \left|\ddot{u}_g(t) + \ddot{x}_{m_2}(t)\right|\right) + 2.0\left(0.25 - \max_t \left|x_{f_2}(t) - x_{m_1}(t)\right|\right) \quad , (22)$$

where $x_{f_i}(t)$ refers to the displacement of i^{th} floor and $\left(x_{f_i}(t) - x_{f_{i-1}}(t)\right)$ is the inter storey drift. $\ddot{u}_g(t)$ is the ground acceleration and $\ddot{x}_{m_2}(t)$ is the acceleration smaller mass block. The displacement $x_{m_1}(t)$ is of the larger mass block, and represents the displacement of the equipment isolation system. The limit state/performance function in Equation 22 is the overall representation of three failure modes. The first term describes the damage to the structural system due to excessive deformation. The second term represents the damage to equipment caused by excessive acceleration. The last term represents the damage of the isolation system. The weighing factors, multiplied with each term in Equation 22, are mainly to emphasize the equal contribution of the individual failure modes to the overall failure of system. It is desirable that (a) inter storey drift is limited to 0.04 *m*, (b) the peak acceleration of the equipment is less than $0.5m/s^2$, and (c) the displacement across the equipment isolation system is less than 0.25 *m*. Equation 22 signifies overall system failure, which does not necessarily occur when above mentioned one or two failure criteria satisfies.



(a)



(b)

Figure 2. Four-storey building (Example 1); (a) Base isolated structure with an equipment isolation system on the second floor; and (b) Acceleration history.

Table 1. Properties of the random variables for Example 1

Random variable	Units	Mean	COV
m_f	kg	6000	0.10
k_f	N/m	30000000	0.10
c_f	$N/m/s$	60000	0.20
f_y	N	20000	0.20
d_y	m	0.05	0.20
k_c	N/m	30000000	0.30
T	s	1.0	0.20
A	$m/m/s$	1.0	0.50

The MPP of the function defined in Equation 22 is obtained using recursive quadratic programming algorithm (Lim and Arora, 1986) with reliability index $\beta = 0.7538$. Five uniformly distributed sample points (n = 5) are deployed along each of the variable axis to estimate HDMR component functions. The reference point is taken as the MPP. Table 2 compares the results obtained by the present method with FORM, SORM, and direct MCS and also presents the computational effort in terms of number of function evaluations, associated with each of the methods.

Table 2. Estimation of failure probability, P_F for Example 1

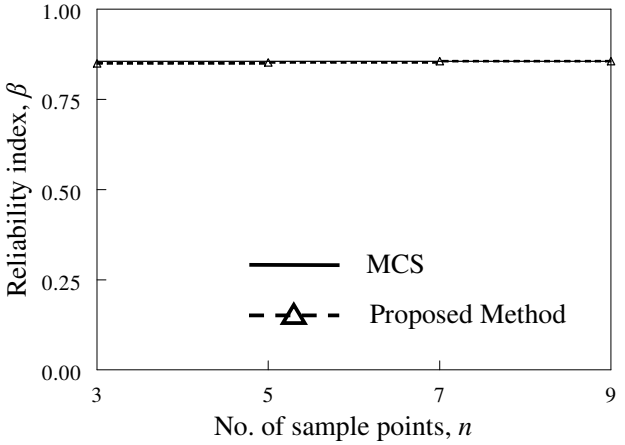
Method	P_F	Number of function evaluation[(a)]
FORM	0.22549	86
SORM	0.21410	356
Direct MCS	0.19599	100000
Present method	0.19696	118[(b)]

(a) Total number of times the original performance function is calculated.

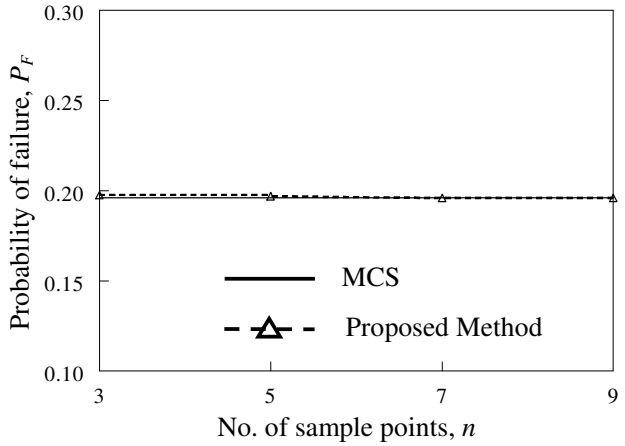
(b) $86 + (n-1) \times N = 86 + (5-1) \times 8 = 118$

The benchmark solution of the failure probability is obtained by direct MCS with $N_S = 10^5$. The COV of P_F corresponding to this sampling size is 0.0064 (computed using Equation 21). Compared with the benchmark solution $(P_F = 0.19599)$, FORM and SORM overestimates the failure probability by around 15.05% $(P_F = 0.22549)$ and 8.46% $(P_F = 0.21410)$, respectively. Proposed method overestimates the failure probability by about 0.49% $(P_F = 0.19696)$ and it needs only 118 (= 86 + 32) function evaluations, while FORM, SORM and direct MCS requires 86, 356 and 10^6 number of original function evaluations, respectively.

To examine the effect of number of sample points used for the present method, similar analyses are carrying out by varying n form 3 to 9. The variation of the reliability index β and the estimated failure probability P_F with respect to number of sample points are presented in Figure 3(a) and Figure 3(b), respectively. It is observed that, P_F obtained using the present method ranges from 0.19769 (+0.86%) (at n = 3) to 0.19598 (−0.01%) (at n = 9). The computational effort in terms of number of function evaluations for the present method are 102, 118, 134 and 150 for n = 3, 5, 7 and 9, respectively. It is observed that n = 5 provides the optimum number of function calls with acceptable accuracy in evaluating the failure probability P_F with the present method.



(a)



(b)

Figure 3. Variation of reliability estimation (Example 1); (a) Reliability index, β; and (b) Probability of failure, P_F

6.2 *Example 2: Three Bay Five Story Frame Structure*

This example examines the performance of the present method for solving reliability problems involving correlated random variables. In this example a three-span, five-story frame structure, studied by Liu and Der Kiureghian (1991), is subjected to horizontal loads, as shown in Figure 4, is investigated.

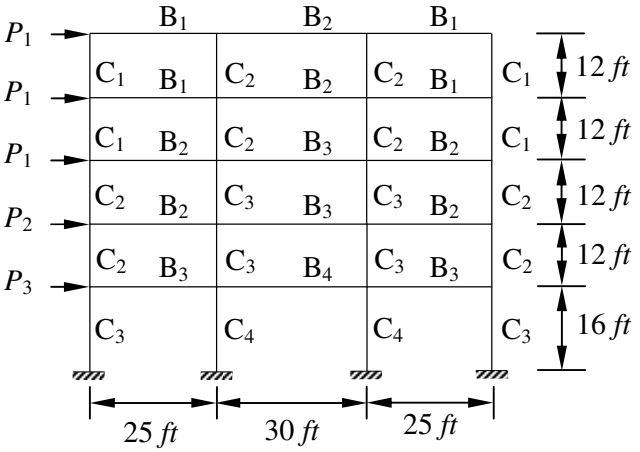


Figure 4. Frame structure for Example 2

There are 21 random variables: (a) three applied loads, (b) two Young's moduli, (c) eight moments of inertia, and (d) eight cross-sectional areas. The random variables associated with frame elements are defined in Table 3. Table 4 lists statistical properties of all random variables. Lateral loads are assumed to be correlated by $\rho = 0.50$. The cross sectional areas and moment of inertias of each beam column elements are highly correlated $(\rho = 0.95)$. Other correlations between cross sectional areas and moment of inertias are assumed as $\rho_{A_iA_j} = \rho_{I_iA_j} = \rho_{I_iI_j} = 0.13$. The elastic modulus are correlated by $\rho = 0.90$. Remaining correlation properties are assumed to be zero.

Table 3. Properties of frame elements for Example 2

Element	Young's modulus	Moment of inertia	Cross-section area
B_1	E_4	I_{10}	A_{18}
B_2	E_4	I_{11}	A_{19}
B_3	E_4	I_{12}	A_{20}
B_4	E_4	I_{13}	A_{21}
C_1	E_5	I_6	A_{14}
C_2	E_5	I_7	A_{15}
C_3	E_5	I_8	A_{16}
C_4	E_5	I_9	A_{17}

Failure criteria is defined as exceedance of the serviceability limit state, i.e., when the horizontal component of the top-floor displacement $u(\boldsymbol{x})$ exceeds 0.2 *ft*, leading to

$$g(\boldsymbol{x}) = 0.2 - u(\boldsymbol{x}). \quad (23)$$

The structure is modeled using ADINA FE software. The MPP of the limit state/performance function defined in Equation 23 is obtained using recursive quadratic programming algorithm (Lim and Arora, 1986) with reliability index $\beta = 3.1599$. Seven uniformly distributed sample points (n = 7) along each of the variable axis are deployed for approximating HDMR component functions. The reference point is taken as the MPP. Table 5 compares the results obtained by the present method with FORM, SORM, and direct MCS. A sampling size $N_S = 10^6$ is considered in direct MCS (Wei and Rahman, 2007) to evaluate the failure probability P_F and the COV of P_F corresponding to this sampling size is 0.053 (computed using Equation 21). Table 5 also contains the computational effort in terms of number of function evaluations, associated with each of the methods. Compared with the failure probability obtained using direct MCS $(P_F = 0.0003630)$, FORM overestimates the failure probability by around 117.12% $(P_F = 0.0007881)$ and SORM underestimates by around 61.38% $(P_F = 0.0001402)$, respectively. The present method overestimates the failure probability by about 2.48% $(P_F = 0.0003720)$. Regarding the computational effort, the present method needs only 600 function evaluations (ADINA FE model run), while FORM, SORM and direct MCS requires 474, 1143 and 10^6 number of original function evaluations respectively. In this aspect, the present method seems slightly computationally expensive than FORM but provides significant accuracy even when compared with SORM.

Table 4. Properties of random variables[(a)] for Example 2

Random variable	Distribution	Mean	Standard deviation
P_1	Rayleigh	30	9
P_2	Rayleigh	20	8
P_3	Rayleigh	16	6.40
E_4	Normal	454000	40000
E_5	Normal	497000	40000
I_6	Normal	0.94	0.12
I_7	Normal	1.33	0.15
I_8	Normal	2.47	0.30
I_9	Normal	3.00	0.35
I_{10}	Normal	1.25	0.30
I_{11}	Normal	1.63	0.40
I_{12}	Normal	2.69	0.65
I_{13}	Normal	3.00	0.75
A_{14}	Normal	3.36	0.60
A_{15}	Normal	4.00	0.80
A_{16}	Normal	5.44	1.00
A_{17}	Normal	6.00	1.20
A_{18}	Normal	2.72	1.00
A_{19}	Normal	3.13	1.10
A_{20}	Normal	4.01	1.30
A_{21}	Normal	4.50	1.50

(a) The units of P_i, E_i, I_i, and A_i are *kip*, *kip/ft*^{2}, *ft*^{4}, and *ft*^{2}, respectively

Table 5. Estimation of failure probability, P_F for Example 2

Method	P_F	Number of function evaluation[(a)]
FORM	0.0007881	474
SORM	0.0001402	1143
Direct MCS [(b)]	0.0003630	1000000
Decomposition Method [(b)]	0.0003829	600
Present Method	0.0003723	600[(c)]

(a) Total number of times the original performance function is calculated.

(b) Wei and Rahman, 2007

(c) $474+(n-1)\times N = 474+(7-1)\times 21 = 600$

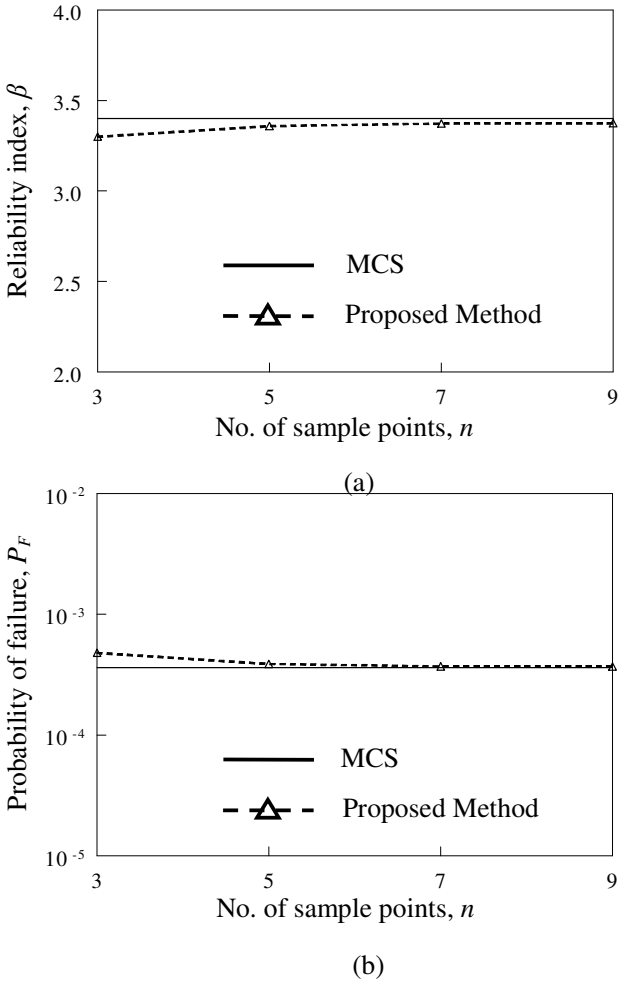


Figure 5. Variation of reliability estimation (Example 2); (a) Reliability index, β; and (b) Probability of failure, P_F

The effect of number of sample points on the present method is examined by carrying a similar analysis varying n form 3 to 9. Figure 5(a) and Figure 5(b) presents respectively, the variation of the reliability index β and the estimated failure probability P_F with respect to number of sample points. Using the present method, P_F varies from 0.000482 (+32.78%) (at n = 3) to 0.000371 (+2.20%) (at n = 9). The computational effort in terms of number of function evaluations for the present method are 512, 558, 600 and 642 for n = 3, 5, 7 and 9, respectively. It is observed that n = 7 provides the optimum number of function calls with acceptable accuracy in evaluating the failure probability P_F with the present method.

7 SUMMARY AND CONCLUSIONS

A new alternative method based on HDMR and FFT to estimate the structural failure probability of structural systems subject to random loads, material properties, and geometry, is presented. The proposed methodology is based on the limit state/performance function approximation and the convolution theorem to estimate the structural failure probability. The limit-state function is obtained by linear approximation of the first-order HDMR component functions at the MPP, and the convolution integral is solved efficiently using the FFT technique. The proposed effort in evaluating the failure probability involves calculating conditional responses at a selected sample points and the FFT of an explicit function. The results of numerical examples indicate that the proposed technique estimates the failure probability accurately with significantly less computational effort compared to the direct MCS. The accuracy and efficiency of the proposed method is demonstrated through numerical examples involving implicit performance functions.

A parametric study is conducted with respect to the number of sample points n used in approximation of HDMR component functions and its effect on the estimated failure probability is investigated. An optimum number of sample points n must be chosen in approximation of HDMR component functions. Very small number of sample points n should be avoided as approximation may not capture the nonlinearity outside the domain of sample points and it affects the estimated failure probability significantly.

8 ACKNOWLEDGEMENT

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